STRENGTH OF MATERIALS

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BY

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PREFACE TO THE THIRD EDITION

No great changes have been made in the third edition. The application of the method of Area Moments has been simplified, the equation for the deflection of a column has been derived by elementary calculus, and slight modifications have been made in the text of many articles. The greater portion of the problems have been rewritten. A few subjects of minor importance have been transferred from the body of the book to the Appendix.

The author is indebted to Professors George E. Beggs, E. H. Wood, P. W. Ott, and E. F. Coddington for suggestions and criticisms.

J. E. B.

Columbus, Ohio,
April, 1924.

PREFACE TO THE SECOND EDITION

In the preparation of the second edition, no radical changes have been made. The most important *addition* is the method of Area Moments for deriving the equations of the elastic line of a beam. This is given along with the usual method of double integration, and the text has been arranged that either one may be studied and the other omitted. A chapter on Curved Beams and Hooks has been added, and a part of a chapter on Theories of Failure.

At the request of a number of teachers, the problems have been largely rewritten. A few more illustrative examples have been worked out in the text at points where experience has shown that the students have difficulty in the application of the theory. Either Cambria Steel or Carnegie Pocket Companion may be used as a handbook with this edition.

The author is under obligation to Professors E. H. Wood, E. R. Maurer, R. N. Menefee, R. W. Gay, and O. H. Basquin, and to Dean F. E. Ayer and Mr. J. O. Draffin, who have kindly furnished suggestions and constructive criticisms.

J. E. B.

COLUMBUS, OHIO,
March, 1917.

PREFACE TO THE FIRST EDITION

This book is intended to give the student a grasp of the physical and mathematical ideas underlying the Mechanics of Materials, together with enough of the experimental facts and simple applications to sustain his interest, fix his theory, and prepare him for the technical subjects as given in works on Machine Design, Reinforced Concrete, or Stresses in Structures.

It is assumed that the reader has completed the Integral Calculus, and has taken a course in Theoretical Mechanics which includes statics and the moment of inertia of plane areas. Chapters XVI and XVII* give a brief discussion of center of gravity and moment of inertia. Students who have not mastered these subjects should study these chapters before taking up Chapter V (preferably before beginning Chapter I).

The problems, which are given with nearly every article, form an essential part of the development of the subject. They were prepared with the twofold object of fixing the theory and enabling the student to discover for himself important facts and applications. The first problems of each set usually require the use of but one new principle,—the one given in the text which immediately precedes; the later problems aim to combine this principle with others previously studied and with the fundamental operations of Mathematics and Mechanics. The constants given in the data or derived from the results of the problems fall within the range of the figures obtained from actual tests of materials. Many of the problems are taken directly from such measurements. Some of them are from tests made by the author or his colleagues at the Ohio State University; others are from bulletins of the University of Illinois Engineering Experiment Station, from "Test of Metals" at the Watertown Arsenal, and from the Transactions of the American Society of Civil Engineers.

This book is designed for use with "Cambria Steel," to which references are made by title instead of by page, so that they are adapted to any edition of the handbook.

*Chapters XIX and XX of the Revised Edition.

The author acknowledges his indebtedness for suggestions and criticisms to Professors C. T. Morris, E. F. Coddington, Robert Meiklejohn, K. D. Swartzel, and many others of the Faculty of the College of Engineering; and to Professor Horace Judd of the Department of Mechanical Engineering for the material for several of the half-tones. He also expresses his obligations to the books which have helped to mold his ideas of the subject,—Johnson's "Materials of Construction," Ewing's "Strength of Materials," and especially the text-books which he has used with his classes,—Merriman's "Mechanics of Materials," Heller's "Stresses in Structures," and Goodman's "Mechanics Applied to Engineering."

The symbols used in the mathematical expressions are much the same as in Heller's "Stresses in Structures."

COLUMBUS, OHIO,
November 6, 1911.

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NOTATION

The symbols which are frequently used in this book are:

a = radius of circle; distance of concentrated load from support.

A, A' = area of cross-section.

b = breadth; breadth of rectangular section; base of triangle; distance of concentrated load from support.

B = some special value of b.

c = distance from neutral axis to extreme fiber; distance of center of curvature of circular beam from center of section; distance in figure.

C = distance from center of curvature of trapezoidal curved beam to intersection of sides.

C_1, C_2, C_3 = integration constants.

d = depth; depth of rectangular section; diameter; distance between parallel axes.

D = some special depth; diameter of boiler.

e = eccentricity of a load on a column; distance in figure.

E = modulus of elasticity.

E_c = modulus of elasticity in compression; modulus of elasticity of concrete.

E_s = modulus of elasticity in shear; tension modulus of elasticity of steel reinforcement.

E_t = modulus of elasticity in tension.

E_v = modulus of volume elasticity.

E_w = working modulus of elasticity.

h = height; distance from vertex to base of triangle.

hp = horsepower.

H = product of inertia.

I = moment of inertia.

I_m = maximum moment of inertia of a beam of variable section.

I_x = moment of inertia with respect to the X axis.

I_y = moment of inertia with respect to the Y axis.

I_o = moment of inertia with respect to an axis through the center of gravity.

j = ratio of moment arm to total depth of a reinforced concrete beam.

J = polar moment of inertia.

k = a constant coefficient; radius of gyration (in Chapter XX); a ratio less than unity.

l = length; length of beam between supports; length of column between points of inflection.

L = length; total length of column.

m = mass of particle; slope of tangent at support; a ratio.

M = moment; mass.

M_o = moment at origin of coördinates.

M_a, M_b, M_c = moment over three consecutive supports.

M_1, M_2, M_3, etc. = moment over first, second, third, etc., supports.
n = ratio, number of turns in a helical spring.
N = normal force at surface; number of revolutions per minute.
p = pitch of rivets; slope of tangent; ratio of steel area to concrete area.
P = concentrated load or force.
q = coefficient in Rankine's formula.
Q = concentrated load or force.
r = distance from origin; radius of gyration (in column formulas); radius.
R = reaction at support; resultant force; radius; radius of coil.
R_1 = reaction at left support; radius of inside surface of curved beam or hook.
R_2 = reaction at second support; radius of outside surface of curved beam or hook.
R_o = radius of neutral surface of curved beam or hook.
s = unit stress.
s_t, s_s, s_c = unit tensile, shearing, and compressive stress.
s_u = ultimate unit stress.
s_w = allowable unit stress.
s' = unit stress resulting from combined shear and tension or compression.
S = unit stress in extreme fibers.
S_1 = unit stress at concave surface of curved beam.
S_2 = unit stress at convex surface of curved beam.
S_s = unit shearing stress at surface of shaft.
t = thickness.
T = torque; tension.
U = work.
U_p = modulus of resilience.
v = distance from neutral axis.
V = total vertical shear.
V_{ab} = total shear near support A in span joining A to B.
w = distributed load per unit of length.
W = total load uniformly distributed.
$\bar{x}$, $\bar{y}$, $\bar{z}$ = coördinates of center of gravity.
y = deflection in a beam or column.
y_{ab} = deflection at B due to a load at A.
y_{max} = maximum deflection in a beam or column.
δ = unit deformation.
δ_s = unit shearing deformation
μ = coefficient of friction.
σ = Poisson's ratio.
ρ = density; radius of curvature.
$\alpha, \beta, \theta, \phi$ = angles in figure.

STRENGTH OF MATERIALS

CHAPTER I

STRESSES

1. Strength of Materials.—That branch of Mechanics which treats of the changes in form and dimensions of elastic solids and the forces which cause these changes is called *The Mechanics of Materials.* When the physical constants and the results of experimental tests upon the materials of construction are included with the theoretical discussion of the ideal elastic solid, the entire subject is called *The Strength of Materials* or *The Resistance of Materials.*

2. Tension.—Support one end of a band of soft rubber, and attach a small hook to the other end, as shown in Fig. 1. Now apply a small weight to the hook. The rubber band is stretched; its length is increased by an amount a, while its cross-section is diminished. Add a second weight. If the second weight is equal to the first one, the elongation b, which it causes, is equal to that caused by the first weight. Remove the weights, and the rubber band returns to its original length and cross-section.

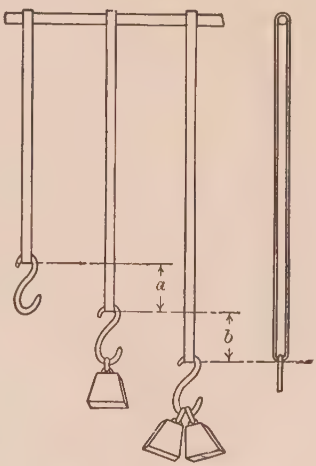


FIG. 1.—Rubber bands in tension.

If steel, iron, wood, concrete, stone, or other solid material is used instead of rubber, the results are similar. There is this apparent difference: while the rubber may be stretched to twice or three times its original length and still return to its original size and shape after the load is removed, one of the other materials may be stretched only a very small amount (usually less than 0.002 of its length), without receiving a permanent change in its dimensions. Again, the force required to produce a relatively

small increase in the length of a rod of wood or steel for instance, is many times greater than that necessary to *double* the length of a soft rubber band of equal cross-section. These differences between the behavior of soft rubber and other solid materials are differences of degree and not of kind. Essentially they are alike.

The rubber bands shown in Fig. 1 are subjected to the action of two forces: the force of the weights pulling downward, and the reaction of the support pulling upward. The bands are in *tension.* A body is said to be in tension when it is subjected to two sets of forces whose resultants are in the same straight line, opposite in direction, and directed *away* from each other.

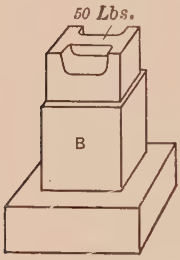


Fig. 2. — Compression.

3. Compression.—When a body is subjected to two sets of forces whose resultants are in the same straight line, opposite in direction, and directed *toward* each other, it is said to be in *compression.* In Fig. 2, the block B is in compression under the action of the 50 pounds pushing down and the reaction of the support pushing up. The effect of compression upon a body is to shorten it in the line of the forces and increase its dimensions in the plane perpendicular to this line.

Tension and compression may be represented as in Fig. 3, in which the arrows represent the forces, and the small rectangles represent the bodies, or portions of a body, upon which the forces act. The rectangles are often omitted; a pair of arrows with their heads together indicate compression, and a pair with their heads in the opposite sense indicate tension.

Fig. 3.

4. Stress; Total Stress.—The force exerted by one body upon another at their surface of contact is called the *stress* between the bodies or the *total stress* between the bodies. If a single body be regarded as cut by an imaginary surface, the force exerted across this surface by either portion of the body upon the other portion is the total *internal stress* in the body at the section. If the forces are such that the portions of the body are pushed together at the imaginary surface, the stress is *compressive.* If the forces tend to pull the portions apart, the stress is *tensile.* Compressive stress at the surface of contact of two separate bodies is called *bearing stress.*

All parts of the bar AB, Fig. 4, are under tensile stress. The total tensile stress at any section CD is the load L and the weight of the hook and of that portion of the bar below the section.

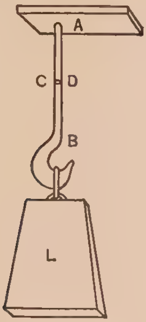


FIG. 4.—Tensile stress.

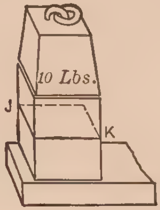


FIG. 5.—Compressive stress.

All parts of the block in Fig. 5 are in compression. The total compressive stress at any section JK is 10 pounds plus the weight of the portion of the block above the section; or, since action and reaction are equal, the total compressive stress is the upward reaction at the base minus the weight of the portion below JK.

5. Unit Stress; Intensity of Stress.—The *unit stress* at any surface is the total stress at the surface divided by its area. Unit stress is frequently called *intensity* of *stress*. In American engineering practice, unit stresses are usually expressed in pounds per square inch. Compressive stresses in masonry are sometimes given in tons per square foot; the bearing presssure of masonry upon soils is always so expressed. English engineers frequently use long tons per square inch to express the intensity of stress in steel and similar solids. Continental engineers,* of course, use kilograms per square centimeter. Physicists, the world over, prefer dynes per square centimeter. In the computation of unit tensile or compressive stresses, the surface considered is a plane section perpendicular to the direction of the forces, unless otherwise stated.

Pounds per square inch are frequently written lb./in.^{2}

Problems

1. The rod AB, Fig. 4, is circular and 2 inches in diameter. If the load L is 42,000 pounds, and the weight of the hook and of the lower part of the rod are neglected, what is the unit stress at any section?

Ans. 13,369 pounds per square inch.

* They sometimes express stress in atmospheres. One atmosphere equals 14.7 lb./in.^{2} equals 1.033 kg./cm.^{2}

2. If the diameter of the rod in Fig. 4 is 3 inches, what must be the load L to produce an intensity of stress of 15,000 pounds per square inch?
3. A pier 18 inches square carries a load of 10,800 pounds. Find the unit compressive stress in the pier in pounds per square inch and the unit bearing stress on the soil at the base in tons per square foot.
Ans. Bearing stress is 2.4 tons per square foot.
4. A short piece of standard 4-inch wrought-iron water pipe, standing on end, supports a load of 26,076 pounds. Find the unit compressive stress. (See handbook for dimensions.) *Ans.* 8,200 pounds per square inch.
5. A short piece of 6-inch by 4-inch by 1-inch angle section, standing on end, supports a load of 111,600 pounds. What is the intensity of the compressive stress? (See handbook.)
Ans. 12,400 pounds per square inch.
6. A short piece of 15-inch, 42-pound I-beam, standing on end, supports a load of 157,250 pounds. What is the unit compressive stress?
Ans. 12,600 pounds per square inch.
7. A piece of 15-inch, 42-pound I-beam, 12 inches long, rests on one flange with the web vertical and supports a load of 73,800 pounds. Find the unit compressive stress in the web.
8. One inch equals 2.540 centimeters and one pound equals 453.6 grams. Find the value of one kilogram per square centimeter in pounds per square inch and compare the result with the handbook.

6. Working Stress; Allowable Unit Stress.—*Working stresses* are the unit stresses to which the materials of a machine or structure are subjected. The *allowable unit stress* for a given material is the maximum working stress which, in the judgment of some authorities, should be applied to that material. For example, the building laws of New York City and the specifications of the American Railway Engineering Association give 16,000 pounds per square inch as the allowable unit tensile stress for structural steel; and the United States Department of Agriculture recommends 1,000 pounds per square inch as the working compressive stress in long-leaf yellow pine, when the load is parallel to the grain and 215 pounds per square inch when the load is across the grain.

Table I gives allowable unit stresses for a few of the most important materials. The values for concrete are taken from the specifications of the Joint Committee on Concrete and Reinforced Concrete. The others are approximately an average of the specifications of the large cities. In order that the student may quickly gain a definite idea of the magnitude of these important stresses, he should memorize this table and the data which follows. In many problems which follow, unless otherwise stated, these allowable stresses will be assumed.

TABLE I.—AVERAGE ALLOWABLE UNIT STRESS

Material	Pounds per square inch	
	Tension	Compression
Structural steel and cast steel	16,000	16,000
Wrought iron	12,000	12,000
Cast iron	3,000	15,000
Long-leaf yellow pine	......	1,200 with the grain
Long-leaf yellow pine	......	250 across grain
White oak	......	1,000 with the grain
White oak	......	400 across grain
Portland cement concrete.....1 : 2 : 4..	......	450
Portland cement concrete ....1 : 3 : 6..	......	300

A steel bar one foot long and one square inch in cross-section weighs 3.4 *pounds.*

One cubic foot of water weighs approximately 62.5 *pounds.*

Problems

(*Use data of Table I unless otherwise specified*)

1. Find the total allowable load, in compression parallel to the grain, which may be applied to a 6-inch by 8-inch short block of long-leaf yellow pine. *Ans.* 57,600 pounds.

2. An eye-bar of structural steel, which exerts a pull of 60,000 pounds, is one inch thick. What should be its width?

3. What must be the dimensions of a cubical block of white oak which supports a load of 40,000 pounds? (Two solutions.)

4. A short 6-inch by 6-inch beam of yellow pine is horizontal and supports a load of 5,000 pounds. The beam is hung on two ¾-inch wrought-iron bolts, each of which carries half the load. Find the maximum unit stress on each bolt. Would ⅝-inch wrought-iron bolts be allowable? If the ¾-inch bolts support the beam by means of nuts and washers, what should be the diameter of each washer? *Ans.* 3.65 inches.

5. Solve Problem 4 for the diameter of the washer if the beam is made of white oak. Solve also for Douglas fir. (See handbook for allowable unit stress.)

6. A short piece of 6-inch wrought-iron water pipe stands on a cast-iron base plate which is supported by a pier of 1:2:4 concrete. If the pipe is subjected to its allowable load, what must be the minimum cross-section of the pier? *Ans.* 149.3 square inches.

7. The pier of Problem 6 is a frustum of a pyramid, 18 inches square at the top, 4 feet square at the bottom, and 9 feet high. The concrete weighs 150 pounds per cubic foot. What is the bearing pressure on the soil at the bottom? *Ans.* Approximately 2.44 tons per square foot.

8. Look up the area of one 10-inch and one 18-inch I-beam and compute the weight of each per foot. Compare with the handbook.
9. A hollow steel cylinder is 8 inches outside diameter, 15 feet long and weighs 1,200 pounds. What is its inside diameter?
10. What is the cross-section of a steel rail weighing 110 pounds per yard?

7. Deformation; Unit Deformation.—The changes in dimensions which occur when forces are applied to a body are called *deformations.* In Fig. 1, the increase in length, a, which takes place when the first load is applied is the deformation caused by that load; the increase b is the deformation caused by the second load; and $a + b$ is the deformation caused by the two loads. The deformation produced by a *tensile* force or *pull* is an *elongation.* The deformation produced by a *compressive* force or *push* is a *compression.* Compression is negative elongation. A deformation which remains after the force is removed is called a *set.*

Unit deformation in a body is the deformation per unit length. In a bar of uniform cross-section, the unit deformation is calculated by dividing the total deformation of a given portion of the bar by the original length of the portion. In Fig. 1, the length a divided by the original length of the band is the unit deformation caused by the first load. Unit deformation is frequently called *relative deformation.*

In algebraic equations many authors represent unit deformation by the letter δ (pronounced delta).

Deformation is frequently called *strain.* The word, *strain,* was formerly used as a synonym for *stress* and is still sometimes heard in that sense. The general practice of technical literature, however, is now to use *strain* to mean *deformation.* When employed in this book it will always have that meaning.

Problems

1. A steel bar is subjected to a tensile stress and a portion of it, originally 8 inches long, is stretched 0.0044 inch. Find the unit elongation.
Ans. 0.00055 inch.
2. A wooden post under compression is shortened 0.144 inch in a length of 15 feet. Find the unit compression. *Ans.* 0.00080.
3. A ¾-inch steel rod, 20 inches long, is subjected to a pull of 6,600 pounds. A portion of the rod, originally 8 inches long, is stretched 0.0041 inch. Find the unit deformation and the unit stress.
4. The coefficient of linear expansion for iron is 0.000012 for degree C. What is the unit deformation and the total deformation in an iron rod 10 feet in length when the temperature changes from 40°F. to 80°F.? when the temperature changes from 50°F. to 30°C.?

8. Elastic Limit.—When a stress is applied to a solid body and then removed, the body returns to its original size and shape, provided the stress has not exceeded a certain limit. If the stress has gone beyond this limit, the body does not return entirely to its original dimensions, but retains some permanent deformation or set. The *unit stress* at this limit is called the *elastic limit* of the material. A wrought-iron rod is stretched about 0.006 inch in a length of 8 inches by a pull of 20,000 pounds per square inch. When the load is removed it returns to its original length. The unit stress of 20,000 pounds per square inch is below the elastic limit of wrought iron. If the load is increased to 30,000 pounds per square inch, the elongation in 8 inches becomes, perhaps 0.075 inch. When this load is removed the rod shortens only 0.009 inch while the remaining 0.066 inch persists as a permanent set. The elastic limit is below 30,000 pounds per square inch.

The elastic limit cannot be determined exactly. A test piece may appear to have no permanent deformation when measured with the usual apparatus and still show some set when more delicate instruments are employed. Time also is a factor. If a load is applied for a considerable period, it causes somewhat greater deformation and considerably greater set than if the time of application is short. Some materials, such as steel, after having been subjected to comparatively large unit stress, frequently show a set of more or less temporary character. When the load is first removed, there is a residual deformation, which may partly or entirely vanish after some little interval.

9. Modulus of Elasticity.—For all stresses below the elastic limit, the unit stress bears a constant ratio to the unit deformation. The quotient obtained by dividing unit stress by the accompanying unit deformation is called the *modulus* of *elasticity* of the material, or *Young's* modulus. In algebraic formulas, modulus of elasticity is represented by the letter E. In algebraic language this definition is

$$E = \frac{s}{\delta}, \qquad \text{Formula I.*}$$

in which E is the modulus of elasticity,

s is the unit stress,

δ is the unit deformation.

*Important formulas, which should be memorized, are designated by Roman numerals in this book.

Problems

1. A steel rod of one square inch cross-section is tested in tension. It is found that a pull of 20,440 pounds stretches 8 inches of the rod 0.0056 inch. Find the unit deformation and the modulus of elasticity.
Ans. Modulus of elasticity is 29,200,000 pounds per square inch.
2. A wooden block, 4 inches square and 16 inches long, is tested in compression. A load of 12,000 pounds shortens 10 inches of the block 0.0060 inch. Find the unit stress, the unit deformation, and the modulus of elasticity.
Ans. Modulus of elasticity is 1,250,000 pounds per square inch.
3. A spruce stick 25 inches long and 1.75 inches square, which was tested in compression at the Bureau of Standards, was shortened 0.01149 inch in a length of 20 inches when the load changed from 612 pounds to 3,672 pounds. Find E for this spruce.
4. A rectangular bar of cold-rolled steel, tested in compression, was 36 inches long, 1.25 inches wide, and 1 inch thick. When the load changed from 625 pounds to 13,125 pounds, the compression in 33 inches was 0.01045 inch. Find E from this reading. *Ans.* $E = 31{,}580{,}000$.
5. A bridge post made of two 12-inch channels, each weighing 30 pounds per foot, is shortened 0.0084 inch in a length of 40 inches by a moving train. If E is 29,000,000 pounds per square inch for this steel, find the additional load on the post when the train passed over the bridge.
Ans. 107,430 pounds.
6. A wrought-iron column, tested at Watertown Arsenal, was 11.31 square inches in cross-section. When the load was changed from 5,000 pounds to 100,000 pounds, the column was shortened 0.0610 inch in a length of 200 inches. Find E for this wrought iron.
7. In a tension test of cast iron at the Watertown Arsenal, an increase of unit stress from 1,000 pounds per square inch to 6,000 pounds per square inch produced an increase in length of 0.0034 inch in a gage length of 10 inches. Find E for this cast iron.

Table II gives values of the modulus of elasticity of a few common materials. This table should be memorized.

TABLE II.—MODULUS OF ELASTICITY

Material	Modulus, in pounds per square inch
Structural steel	29,000,000
Hard steel	30,000,000
Wrought iron	27,00,0000
Cast iron	15,000,000
Timber (parallel to the grain)	1,000,000 to 2,000,000
Portland cement concrete	2,000,000 to 4,000,000

8. A bar of cast iron, 3 inches by 1 inch, is shortened 0.0056 inch in a length of 8 inches. Find the load applied. *Ans.* 31,500 pounds.

9. A structural steel bar, weighing 6.8 pounds per foot, is stretched 0.0128 inch in a length of 20 inches. Find the total pull.

Ans. 37,120 pounds.

10. A 6-inch by 6-inch by 1-inch angle, 4 feet in length, is subjected to a pull of 176,000 pounds. Find the elongation in a length of 30 inches.
11. The temperature coefficient of steel is 0.0000067 per degree Fahrenheit. How much stress is developed in a rod of structural steel when the temperature changes from 80°F. to 20°F. and the rod is not allowed to contract?
12. A foreign handbook gives the modulus of elasticity in kilograms per square millimeter as follows:

Steel	20,400
Copper (drawn)	12,400
Brass	10,800
Aluminum (drawn)	7,500

Reduce these to pounds per square inch.

10. Physical Meaning of E.—Formula I of Article 9 may be written

$$\delta = \frac{s}{E}.$$

If s be made equal to unity, δ becomes equal to $\frac{1}{E}$. With the common engineering units, the reciprocal of E is the unit deformation produced by a unit load of one pound per square inch.

The modulus of elasticity of steel in tension is about 30,000,000 pounds per square inch. This means that a pull of one pound on a bar one inch square will stretch every inch of this bar one thirty-millionth of an inch. A pull of 1,000 pounds applied to a bar one inch square will stretch every inch of its length one thirty-thousandth of an inch. If the elastic limit is not exceeded, a pull of 30,000 pounds per square inch of cross-section will stretch each inch of the bar one thousandth of an inch.

Examples

Solve without writing

1. A steel rod, one inch square and 30 inches long, is subjected to a pull of 15,000 pounds. If E is 30,000,000 pounds per square inch, what is the stretch per inch of length? What is the total elongation in a length of 20 inches?
2. If wood for which the modulus of elasticity is 1,500,000 pounds per square inch is subjected to a compressive load of 600 pounds per square inch, what is the compression per inch of length? What is the total compression in a length of 10 feet?

3. A 2-inch by 4-inch wooden block is subjected to a compressive load of 3,600 pounds. If the moduls of elasticity parallel to the fibers is 1,200,000 pounds per square inch, what is the total compression in a length of 5 feet?
4. What is the unit elongation in cast iron when the unit tensile stress is at its allowable limit? What is the unit compression when the unit compressive stress is at its allowable limit?

Formula I may also be written

$$s = E\delta,$$

which defines E as the coefficient to be multiplied into the unit deformation to obtain the unit stress. A deformation of one one-thousandth of an inch per inch of length is generally not far from the elastic limit. Since many micrometers measure in thousandths of an inch, this length has a definite meaning to all persons who do exact mechanical work or make precise measurements. It is desirable, therefore, to fix the attention on the unit stress which accompanies a unit deformation of one one-thousandth of the original length, or, expressed in a slightly different way, on the unit stress which accompanies a relative deformation of one-tenth of one per cent.

Examples

Solve without writing

5. What is the unit stress when the unit deformation is 0.001 for structural steel, for wrought iron, and for cast iron?
6. If the modulus of white oak is 1,600,000 pounds per square inch, what is the total load on a 2-inch by 4-inch piece which is shortened 0.0064 inch in a length of 16 inches?
7. A rod of hard steel, 3 feet in length, weighs 30.6 pounds. The bar is shortened 0.0080 inch in a length of 20 inches. What is the total load?

If in Formula I δ is made equal to unity, s becomes equal to E. From this relation the modulus of elasticity in tension is sometimes defined as the unit stress which would double the length of a rod of uniform cross-section, if such doubling were possible without breaking the rod or exceeding the elastic limit.

11. Work and Resilience.—When a force acts on a body and motion takes place in the direction of the force, the force is said to do *work*. The distance which the point of application moves is called the *displacement*. The work done by a constant force is the product of the force multiplied by the displacement. If P represents the constant force and x represents the displacement of

its point of application, the work is the product Px, provided the force and displacement are in the same direction. If the force is in pounds and the displacement is in feet, the work is expressed in *foot-pounds*. If the force is not constant, the work is the product of the *average force* multiplied by the displacement. When an elastic body is deformed, the force varies directly as the displacement (provided the elastic limit is not exceeded) and the average force is half the sum of the initial and final forces.

Problems

1. A given spring is stretched one inch by a load of 12 pounds. What force will stretch this spring 3 inches? What is the average force when the spring is stretched 3 inches? What is the work done in stretching the spring 3 inches? *Ans.* 36 pounds; 18 pounds; 54 inch-pounds.

2. After the spring of Problem 1 has been stretched 3 inches, an additional force is applied, which produces an additional elongation of 4 inches. What is this additional force? What is the average force while the spring is stretched the last 4 inches? What is the work done in stretching the spring the last four inches?
Ans. 48 pounds; 60 pounds; 20 foot pounds.

3. The spring of Problem 1 is stretched 7 inches from the position of zero elongation. Find the final pull, the average force, and the total work. Check the total work by comparison with the results of Problems 1 and 2.

4. A load of 24,000 pounds is applied to a steel rod which has no initial load. The elongation is 0.03 inch. Find the work in foot-pounds.
Ans. 30 foot-pounds.

5. A pull of 60,000 pounds is applied to a steel rod of 6 square inches cross-section. If the modulus of the steel is 30,000,000 pounds per square inch, what is the work done in a length of 20 feet? *Ans.* 200 foot-pounds.

6. What would be the work in Problem 5 if the load were applied to a rod of 3 square inches cross-section and 20 feet length?

7. The allowable unit stress in compression is applied to a cast-iron bar which is 2 inches square and 12 inches long. Find the total work in foot-pounds. Find the work in inch-pounds per cubic inch.

The work done in deforming a body is stored up as elastic potential energy, which may be recovered as mechanical work when the load is removed. This elastic energy is called the *resilience* of the material. If the unit stress does not exceed the elastic limit, practically all the work which is put into the body may be recovered. If the stress exceeds the elastic limit, part of the energy is converted into heat and can not be directly regained as mechanical work.

12. Modulus of Resilience.—The work expended in deforming unit volume of any material to the elastic limit is called the

modulus of resilience of the material. It is the *elastic potential energy* of unit volume when stressed to the elastic limit. The modulus of resilience is a *measure* of the amount of energy which may be stored in a given material and recovered as mechanical work without loss.

If unit volume of a solid is subjected to unit stress s, the deformation in unit length is $\frac{s}{E}$ and the total work is

$$U_p = \frac{s}{2} \times \frac{s}{E} = \frac{s^2}{2E}. \qquad \text{Formula II.}$$

This expression $\left(\text{energy in unit volume} = \frac{s^2}{2E}\right)$ gives the energy for any value of s below the elastic limit. When s is the unit stress at the elastic limit, the expression is the modulus of resilience. When s and E are given in pounds per square inch, Formula II gives the energy in *inch-pounds* per *cubic inch*.

The total elastic energy in a body, all parts of which are subjected to a unit stress s, is obtained by multiplying the total volume of the body by the energy per unit volume, and is independent of the form of body.

Problems

1. Find the modulus of resilience of steel for which the modulus of elasticity is 30,000,000 pounds per square inch and the elastic limit is 36,000 pounds per square inch. *Ans.* 21.6 inch-pounds per cubic inch.
2. Find the modulus of resilience of spring steel for which E is 30,000,000 pounds per square inch and the elastic limit is 96,000 pounds per square inch. *Ans.* 153.6 inch-pounds per cubic inch.
3. What is the modulus of resilience of timber for which the modulus of elasticity is 1,600,000 pounds per square inch and the elastic limit is 3,600 pounds per square inch?
4. How high can a mass of the material of Problem 2 be lifted by the energy which is stored in it at the elastic limit? *Ans.* 45.2 feet.
5. If a cubic foot of the timber of Problem 3 weighs 36 pounds, how high will the energy stored in a piece of this timber lift its own weight?
6. What is the energy in a cubic inch of cast iron when subjected to its allowable unit stress? What is the total energy in a bar 2 inches square and 12 inches long?
7. A steel rod, 2 inches in diameter and 20 inches long, is subjected to a compressive load of 60,000 pounds and is shortened 0.0128 inch. Find the total work done by the load and then find the work per unit volume. Check the work per unit volume by Formula II.

The work of resilience has been calculated by multiplying the average force by the deformation. The same result may be

obtained by integration. Let x represent the total elongation of a rod of length l and unit cross-section; and let dx represent an infinitesimal increment of this elongation. The unit elongation is $\frac{x}{l}$ and the unit stress $\frac{Ex}{l}$. The work done in causing a deformation dx in a rod of unit cross-section is the unit stress, which is regarded as constant for the infinitesimal displacement, multiplied by dx.

$$\text{Increment of work} = \frac{Ex}{l}\,dx. \tag{1}$$

$$\text{Total work} = \int \frac{Ex}{l}\,dx = \frac{E}{2l}\Big[x^2\Big]_{x_1}^{x_2} = \frac{E}{2l}(x_2^2 - x_1^2), \tag{2}$$

in which x_1 is the initial deformation and x_2 is the final deformation. The unit stress when the deformation is x in a length l is $s = \frac{Ex}{l}$, from which $x = \frac{sl}{E}$. If the values of x_1 and x_2 in Equation (2) are expressed in terms of the corresponding stresses,

$$\text{Total work} = l\left(\frac{s_2^2 - s_1^2}{2E}\right) = \left(\frac{s_2^2 - s_1^2}{2E}\right) \times \text{volume}. \tag{3}$$

If the initial stress is zero, equation (3) becomes Formula II.

Problems

8. Derive Equation (3) by means of average force without integrating.

9. Find the work done in 200 cubic inches of cast iron in compression when the unit stress changes from 0 to 6,000 pounds per square inch and when it changes from 6,000 to 12,000 pounds per square inch.

13. Poisson's Ratio.—A body which is subjected to tensile stress is elongated, and the amount of the elongation, provided the elastic limit is not exceeded, is proportional to the unit stress. At the same time the diameter of the body becomes smaller. A body subjected to compressive stress is shortened in the direction of the load while its transverse dimensions are increased. The ratio of the unit deformation at right angles to the direction of the load to the unit deformation in the direction of the load is called Poisson's ratio. The value of this ratio varies with the material. Usually the ratio is slightly greater than one-fourth. For steel it is about 0.27. If a rod of steel is elongated 0.001 of its length, its diameter is reduced about 0.00027 of its original value. Poisson's ratio will be represented in this book by the Greek letter σ* (sigma).

* There is no definite agreement as to the symbol for Poisson's ratio. Some writers use $\frac{1}{m}$, others use σ or ρ or μ.

Problems

1. A steel rod, 3 inches in diameter, is stretched 0.0064 inch in a length of 8 inches. At the same time the diameter is reduced 0.00065 inch. Find Poisson's Ratio. *Ans.* 0.27.
2. If Poisson's ratio is 0.27 and the modulus of elasticity is 29,000,000, how much is the width of a 5-inch by 2-inch steel bar decreased by a pull of 174,000 pounds? *Ans.* 0.00081 inch.
3. In Problem 2, if the unit stress is proportional to the unit deformation, what is the transverse unit compressive stress? *Ans.* 4,700 pounds per square inch.
4. Poisson's ratio for copper is about 1/3 and the modulus of elasticity is 16,000,000 pounds per square inch. How much is the diameter of a 2-inch round copper rod increased when subjected to a compressive load of 30,000 pounds? *Ans.* 0.000398 inch.
5. A steel plate is subjected to a tensile stress of 12,000 pounds per square inch parallel to the X axis and a tensile stress of 6,000 pounds per square inch parallel to the Y axis. If E is 30,000,000 pounds per square inch and Poisson's ratio is 1/4, what is the unit deformation in the direction of each coördinate axis?

	Axis	Unit deformation
Ans.	X	0.00035 elongation.
	Y	0.00010 elongation.
	Z	0.00015 compression.

6. Solve Problem 5 if the unit stress along the Y axis is compression.

Ans.	X	0.00045 elongation.
	Y	0.00030 compression.
	Z	0.00005 compression.

When *biaxial* loads are applied, as in Problems 5 and 6, the unit deformation may be greater or less than that caused by a single load. In Problem 6, the tensile stress of 12,000 pounds per square inch in the direction of the X axis produces unit elongation of 0.00040 in that direction. The compressive stress of 6,000 pounds per square inch along the Y axis produces unit compression of 0.00020 in the direction of that axis. Since Poisson's ratio is 1/4, the compression of 0.00020 along the Y axis causes an elongation of 0.00005 along the X axis; so that the total unit elongation along the X axis becomes 0.00045. According to *St. Venant's law** the entire unit deformation in any direction caused by any combination of forces should not exceed the unit deformation which would be produced by the allowable unit stress. In Problem 6, the entire unit elongation in the direction of the X axis is equivalent to the unit deformation which

* St. Venant's law and other theories in regard to the allowable unit stress are discussed further in Chapter XVII.

would be caused by a stress of 13,500 pounds per square inch along that axis.

If, in Problem 6, the unit compressive stress parallel to the Y axis were 12,000 pounds per square inch instead of 6,000 pounds per square inch, the entire unit elongation along the X axis would be equivalent to that caused by $12{,}000 + \frac{12{,}000}{4}$ or 15,000 pounds per square inch.

The unit stress which is equivalent to that which will produce a deformation equal to the deformation caused by a combination of stresses, may be calculated directly by simply multiplying each unit stress by Poisson's ratio and adding it, with the proper sign, to the other stresses.

Example

A block of metal is subjected to a compressive stress of 8,000 pounds per square inch parallel to the X axis, a tensile stress of 6,000 pounds per square inch along the Y axis, and a compressive stress of 5,000 pounds per square inch along the Z axis. Find the unit stress along each axis which will be equivalent to the stress which gives the deformation which is given by this combination.

Ans. X axis—Equivalent unit stress $= 8{,}000 + 1{,}500 - 1{,}250 = 8{,}250$ lb./in.^{2}

Problems

7. A rod of material, for which Poisson's ratio is ¼ and the allowable unit tensile stress is 1,800 pounds per square inch, is subjected to a transverse compression of 2,400 pounds per square inch and a pull in the direction of its length. What is the maximum allowable pull?

Ans. 1,200 pounds per square inch.

8. The rod of Problem 7 is subjected to a transverse compression of 2,000 pounds per square inch in one direction and a second transverse compression of 1,200 pounds per square inch at right angles to the first. Find the maximum allowable pull in a direction at right angles to the plane of these compressive stresses.

The effects of biaxial loading are plainly seen in the behavior of brittle materials under tensile tests. In order to test a solid in tension, the ends must be held by grips which cause transverse compression. Suppose a porcelain test bar is 1 inch square at the middle and 1 inch by 1.5 inches at the heads. Suppose that the transverse stress at the heads necessary to hold the bar is 8,000 pounds per square inch and that the total tension is 3,000 pounds. The direct tensile stress in the heads is 2,000 pounds per square inch and the tensile stress caused by the compression

(if Poisson's ratio is one-fourth) is an equal amount. The total tensile stress in the heads is then 4,000 pounds per square inch, while the unit tensile stress at the middle, which is supposed to be the weakest, is only 3,000 pounds per square inch. Such bars usually fail at the heads. It is very difficult to develop the full tensile strength of porcelain or other very brittle material.

14. Volume Change and Volume Modulus of Elasticity.—When a solid is subjected to stress in one direction, there is a slight change in volume. If the load is compressive, the length becomes less and the area becomes greater. If the load is tensile, the length becomes greater and the area becomes less.

Problems

1. A steel bar, 2 inches square and 10 inches long, is subjected to a compressive load of 60,000 pounds in the direction of its length. If E is 30,000,000 pounds per square inch and Poisson's ratio is 0.28, what is the length, the area of cross-section and the volume of the bar under the load?
Ans. 9.995 inches; 4.00112 square inches; 39.9912 cubic inches.
2. A steel rod, for which E is 30,000,000 pounds per square inch, is 2 inches in diameter. How much is the volume of 20 inches of this rod increased by a pull of 90,000 pounds, if Poisson's ratio is 0.27?

If a unit cube is elongated an amount δ by an external pull, its length becomes $1 + \delta$ and its transverse dimensions become $1 - \sigma\delta$ in which σ is Poisson's ratio.

$$\text{Area of cross-section} = (1 - \sigma\delta)^2 = 1 - 2\sigma\delta + (\sigma\delta)^2. \quad (1)$$

Since $\sigma\delta$ is small, being never greater than 0.001, its square, which is relatively much smaller, may be neglected, so that, approximately,

$$\text{Area of cross-section} = 1 - 2\sigma\delta \quad (2)$$

Multiplying by the length,

$$\text{Volume} = (1 - 2\sigma\delta)(1 + \delta) = 1 + (1 - 2\sigma)\delta - 2\sigma\delta^2, \quad (3)$$

of which the last term, $2\sigma\delta^2$, may also be neglected, so that

$$\text{Approximate volume} = 1 + (1 - 2\sigma)\delta. \quad (4)$$

Subtracting the original volume of one cubic unit,

$$\text{Increment of volume} = (1 - 2\sigma)\delta. \quad (5)$$

These formulas apply only to the temporary deformations below the elastic limit. For the permanent change of form which occurs

when the elastic limit is exceeded there is practically no change of volume.

Problems

3. A 2-inch cube is stretched 0.0012 inch. If Poisson's ratio is 0.26, how much is the area of cross-section reduced, and how much is the volume increased? Solve without formulas and check by Equations (2) and (5).
Ans. 0.001248 square inch reduction of area; 0.002304 cubic inch increase of volume.

4. A steel bar 2 inches square is subjected to a compression of 120,000 pounds. If E is 30,000,000 pounds per square inch and Poisson's ratio is 0.28, find the reduction of volume per cubic inch and the total reduction in a length of 10 inches.

A solid submerged in a liquid is under pressure from all directions. The quotient obtained when the unit pressure is divided by the relative reduction of volume is called the *modulus of volume elasticity*. If, for instance, one cubic inch of a solid is reduced to 0.9995 cubic inch by a pressure of 10,000 pounds per square inch in all directions, the modulus of volume elasticity is

$$E_v = \frac{10{,}000}{0.0005} = 20{,}000{,}000 \text{ pounds per square inch.}$$

Problems

5. A block of steel has its volume changed from 8.240 cubic inches to 8.236 cubic inches by a pressure of 10,000 pounds per square inch. Find the modulus of volume elasticity. *Ans.* 20,600,000 pounds per square inch.

6. A copper cylinder, for which E_v equals 16,000,000 pounds per square inch, is sunk in water to a depth of 25,000 ft. How much is its volume reduced?

The modulus of volume elasticity may be computed from the modulus of linear elasticity (Young's modulus) and Poisson's ratio. If a cube of unit dimensions is subjected to unit pressure s in the direction of any axis, it is shortened $\frac{s}{E}$ in the direction of the pressure and elongated $\frac{\sigma s}{E}$ along each of the two axes at right angles to the direction of the pressure. When there is a compressive stress s in every direction, the compression along any axis is made up of the direct compression $\frac{s}{E}$, which is due to the pressure in that direction and two elongations $\frac{\sigma s}{E}$, each of which is

due to pressure along one of the other axes at right angles to the first.

$$\text{Total compression} = \frac{s}{E} - \frac{2\sigma s}{E} = \frac{s}{E}(1 - 2\sigma). \qquad (6)$$

The length of each edge of the cube becomes $1 - \frac{s}{E}(1 - 2\sigma)$, and

$$\text{final volume} = \left\{1 - \frac{s}{E}(1 - 2\sigma)\right\}^3 = 1 - \frac{3s}{E}(1 - 2\sigma) + \frac{3s^2}{E^2}(1 - 2\sigma)^2 +, \text{etc.} \qquad (7)$$

Since $\frac{s}{E}$ is very small, the terms containing the higher powers may be dropped and Equation (7) then becomes

$$\text{final volume} = 1 - \frac{3s}{E}(1 - 2\sigma). \qquad (8)$$

Since the original volume was unity, the decrease of volume is $\frac{3s}{E}(1 - 2\sigma)$. The modulus of volume elasticity is obtained by dividing the unit stress s by the reduction of unit volume.

$$E_v = \frac{s}{\frac{3s}{E}(1 - 2\sigma)} = \frac{E}{3(1 - 2\sigma)} \qquad (9)$$

Problems

7. Show that the modulus of volume elasticity is two-thirds the modulus of linear elasticity if Poisson's ratio is one-fourth.

8. If E for hard steel is 30,000,000 pounds per square inch and E_v is 22,200,-000 pounds per square inch, what is Poisson's ratio?

9. In a test made by E. H. Amagat (*Annales de Chimie et de Physique*, 1891, pages 95–140) a glass tube, 1 meter long, was placed inside a strong cylinder and subjected to liquid pressure in all directions. When the pressure was changed 500 atmospheres the length of the tube was reduced 0.000375 meter. If the unit deformation was the same in all directions, what was the unit volume deformation? If one atmosphere equals 14.7 pounds per square inch, find the modulus of volume elasticity. For similar glass E was found to be 10,280,000 pounds per square inch. Find Poisson's ratio. *Ans.* E_v = 6,533,000 pounds per square inch; Poisson's ratio = 0.238.

Miscellaneous Problems

1. A stick of Douglas fir tested in tension at the Watertown Arsenal ("Tests of Metals," 1896, page 405) showed an elongation of 0.0427 inch in a gage length of 200 inches when the load per square inch changed from 100 pounds to 500 pounds. Find E.

Ans. 1,874,000 pounds per square inch.

2. A second stick of Douglas fir tested in tension ("Tests of Metals," 1896, pages 407–09) showed an elongation of 0.1015 inch in a gage length of 200 inches, and a decrease of width of 0.0020 inch in a width of 12 inches when the load changed from 100 pounds to 1,000 pounds per square inch. Find the modulus of elasticity in tension parallel to the grain and Poisson's ratio. *Ans.* Poisson's ratio, 0.33.

3. In a compressive piece cut from the stick of Problem 2, when the compressive stress changed from 100 pounds to 1,000 pounds per square inch, there was a compression of 0.0230 inch in a gage length of 50 inches. Find E_c.

4. A white-oak stick 11.98 inches by 9.95 inches tested in compression ("Tests of Metals," 1896, page 425) was shortened 0.0140 inch in a gage length of 50 inches when the load was increased from 11,920 pounds to 71,520 pounds. Find E.

5. A block of the same oak used in Problem 4 was tested in compression across the grain. When the unit stress changed from 20 pounds per square inch to 320 pounds per square inch, the compression in a gage length of 6 inches was 0.0091 inch. Find the modulus of elasticity of oak across the grain. *Ans.* 198,000 pounds per square inch.

6. Two blocks of Douglas fir were tested in compression across the grain. In the first block the compression was normal to the growth rings, and the compression in a gage length of 6 inches when the unit load changed from 20 pounds to 300 pounds was 0.0081 inch. In the second block the compression was tangent to the growth rings, and the compression in 6 inches with the same change of load was 0.0195 inch. Find E for each case ("Tests of Metals," 1896, pages 396–97).

7. A built I-section column tested at Watertown Arsenal (1909, page 779) was 3 feet 5.25 inches long and had a sectional area of 13.85 square inches. A gage length of 20 inches was shortened 0.0091 inch when the total load changed from 13,850 pounds to 207,750 pounds, and was shortened 0.0129 inch when the load changed from 1,000 pounds per square inch to 20,000 pounds per square inch. Find E from each set of measurements.

8. A test bar made from a steel rail (Watertown Arsenal, 1909, page 898) was 0.798 inch in diameter. The elongation in a gage length of 10 inches when the load changed from 500 pounds to 25,500 pounds was 0.0177 inch. Find E for this steel.

9. A steel bar in the form of a frustum of a pyramid is 1 inch square at one end, 2 inches square at the other end, and 10 inches long. A load of 30,000 pounds is applied in compression. If E is 30,000,000 pounds per square inch, and if it is assumed that the stress in any transverse section is uniform throughout the section, calculate the decrease in length by means of integral calculus. Compare with a uniform bar 1.5 inches square. *Ans.* Total compression is 0.005 inch.

10. By integration find the total internal work of the bar of Problem 9. *Ans.* 75 inch-pounds.

11. The total external work of the bar of Problem 9 is the product of the total compression multiplied by the average load. Solve for the external work by means of the answer of Problem 9 and compare with the internal work.

12. A bar 1 inch thick and 20 inches long is 1 inch wide at one end and increases uniformly to a width of 3 inches at the other end. If a load of 30,000 pounds in compression is applied to the bar, find the unit stress at a distance x from the small end. If E is 30,000,000 pounds per square inch, find the unit deformation, and find the total deformation in the entire length. *Ans.* 0.010986 inch.

13. By integration solve Problem 12 for the total internal work and then check Problem 12 by means of the external work.

14. A plate of uniform thickness t has a breadth b at one end of a given length l and a breadth c at the other end. Find the expression for the elongation of this length l due to a pull P. *Ans.* $\dfrac{Pl}{Et(c-b)} \log \dfrac{c}{b}$.

15. An oak block, 6 inches square and 30 inches long, is bolted between two steel plates, each 6 inches wide, ½ inch thick, and 30 inches long. A force applied lengthwise the combined block shortens it 0.008 inch in a length of 20 inches. If E for the steel is 29,000,000 pounds per square inch and E for the oak is 1,600,000 pounds per square inch, what is the total force? *Ans.* 92,640 pounds.

16. In Problem 15, what is the unit stress in the steel when the unit stress in the oak is at its allowable value in compression.

17. A vertical pier, 26 inches square, is made of concrete in which four 4-inch by 3-inch by ½-inch structural steel angles are imbedded. The modulus of elasticity of the steel is fifteen times the modulus of the concrete. When the pier carries a load of 360,360 pounds, how much of this load is carried by the concrete and how much is carried by the steel? What is the unit stress in each?

18. A hollow steel cylinder, 1 inch inside diameter and 2 inches outside diameter, is supported in a vertical position by three small lugs which do not appreciably change the cross-section. A 1-inch wrought-iron bolt passes through this cylinder. A nut near the bottom of the bolt is turned up against the lower end of the cylinder until the tensile stress in the bolt reaches 10,000 pounds per square inch. If E is 30,000,000 pounds per square inch for the steel and 27,000,000 pounds per square inch for the wrought iron, what is the unit stress in each? A load of 6,000 pounds is then hung on the lower end of the bolt and supported by a second nut which does not touch the nut in contact with the cylinder. If the deformation of the upper nut and of the part of the bolt which passes through it be neglected, and if the deformation of the head of the bolt also be neglected, what is the total tension and what is the unit tensile stress in the bolt if the supporting lugs are at the top of the hollow cylinder? *Ans.* Total, 9,239 pound tension in bolt.

19. Solve Problem 18 if the supporting lugs are at the bottom of the cylinder.

CHAPTER II

STRESS BEYOND THE ELASTIC LIMIT

15. Stress-strain Diagrams.—In Chapter I the only unit stresses considered are below the elastic limit. Within that limit unit stress is proportional to unit deformation, and Formula I and the equations of Article 14 hold good. Unit stresses below the elastic limit are the most important from the standpoint of the engineer, for in well-designed structures the unit stress seldom exceeds one-half of this limit. It is desirable, however, to know what takes place above the elastic limit and the character of the final failure of the material. To secure this information, tests are made in which a series of loads are applied to a piece of the material in question, and the corresponding deformations are observed with suitable measuring apparatus. Table III gives a part* of the results of a tension test of a rod of† machine steel. The rod was originally 20 inches long and turned to a diameter of 1.31 inches. About 9 inches of the rod at the middle was turned down further to a diameter of 1.115 inches. A length of 8 inches in this middle portion was taken as the gage length from which to measure elongations. The rod I on the right in Fig. 6 (photographed from a rod exactly like the one tested) shows the original form of this test piece. The elongations in this gage length were measured by an extensometer reading to 0.0001 inch (see Johnson's "Materials of Construction," Fig. 271). As there are two micrometers in this extensometer, the gage readings are given to one-half of a division. For instance, when the load was 5,000 pounds per square inch, the sum of the readings was 0.0029 inch. The elongation in the gage length is given as 0.00145. The accidental error of reading may easily

* Readings were taken at 2,000-pound intervals from 56,000 to 76,000 pounds per square inch, and were used in locating the curve of Fig. 6. Readings were also taken at 2,000-pound intervals between 30,000 and 40,000 pounds per square inch, as it was suspected that the yield point might fall between these limits.

† An analysis of this steel, made by Prof. D. J. Demorest, gave: carbon, 0.42 of 1 per cent.; manganese, 0.71 of 1 per cent. The rod was turned from a bar of hot-rolled steel.

TABLE III.—TENSION TEST OF MACHINE STEEL

Diameter, 1.115 inches; area of section, 0.976 square inch; gage length, 8 inches

Applied load		Elongation	
Total	Per square inch	In gage length	Per inch length
Pounds	Pounds	Inch	Inch
0	0	0	0
2,926	3,000	0.00085	0.00011
4,880	5,000	0.00145	0.00018
9,760	10,000	0.00260	0.00033
14,640	15,000	0.00410	0.00051
19,520	20,000	0.00535	0.00067
24,400	25,000	0.00665	0.00083
29,280	30,000	0.00795	0.00099
34,160	35,000	0.00920	0.00115
39,040	40,000	0.01075	0.00134
40,992	42,000	0.0114	0.00142
42,944	44,000	0.0144	0.00180
44,896	46,000	0.0356	0.00445
44,000	45,080	0.0734	0.00917
44,500	45,504	0.0965	0.01206
45,000	46,100	0.0973	0.01216
45,872	47,000	0.0981	0.01226
46,848	48,000	0.0991	0.01239
47,824	49,000	0.1013	0.01266
48,800	50,000	0.1163	0.01454
50,752	52,000	0.1273	0.01589
52,704	54,000	0.1381	0.01726
54,656	56,000	0.1552	0.01940
64,416	66,000	0.2601 (1)*	0.03251
74,176	76,000	0.4244 (2)	0.05305
76,128	78,000	0.50 (by	0.0625
78,080	80,000	0.59 scale)	0.0740
79,056	81,000	0.70	0.0875
80,032	82,000	0.76	0.095
81,008	83,000	0.85	0.106
81,984	84,000	0.99	0.124
83,000	85,040	1.24	0.155
83,200	85,240	1.50	0.187
82,000	84,100	1.64 (3)	0.205
80,000	82,000	1.85 (4)	0.231
72,000	73,800 (broke)	1.99 (5)	0.247

* (1) Diameter, 1.097 inches. (2) Diameter, 1.083 inches.
(3) Begins to "neck." (4) Diameter of neck, 0.904 inch.
(5) Elongation measured after fracture. Diameter of neck, 0.821 inch.
Steel, hot-rolled; carbon, 0.42 per cent.

be as great as one part in 29, instead of one part in 290 as the figures might be assumed to indicate.

When the load reached 78,000 pounds per square inch, the extensometer was removed and the elongations were taken with an ordinary steel scale reading in hundredths of an inch. After the rod was broken, it was taken from the testing machine. The two portions were placed together as shown in Fig. 6, II; and the final elongation of 1.99 inches was measured. Loads were applied and measured by means of a 100,000-pound testing machine. (See Moore's Materials of Engineering, Fig. 63, or Hatt and Scofield's Laboratory Manual of Testing Materials, Fig. 12.)

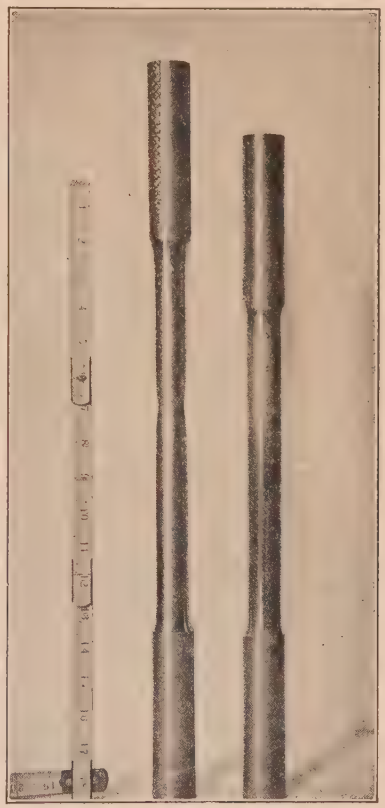


FIG. 6.—Steel rod tested in tension.

In order to present the results of such a test visually, it is convenient to use the unit stress and the unit elongation as the coördinates in a curve called the *stress-strain diagram*, or simply *stress diagram*.

In America, the unit stress in pounds per square inch is used as ordinate, and the unit deformation is taken as abscissa. In England, some writers use unit stress as abscissa and unit deformation as ordinate.

Figure 7 is the stress-strain diagram plotted from Table III. One division on the horizontal scale represents a unit elongation of 0.01, and one division on the vertical scale represents a unit stress of 5,000 pounds per square inch.

Figure 8 is a part of the stress-strain diagram from the same table plotted on an enlarged scale; one division on the horizontal represents a unit elongation of 0.0002 inch per inch of length (one-fiftieth as much as in Fig. 7); one division on the vertical

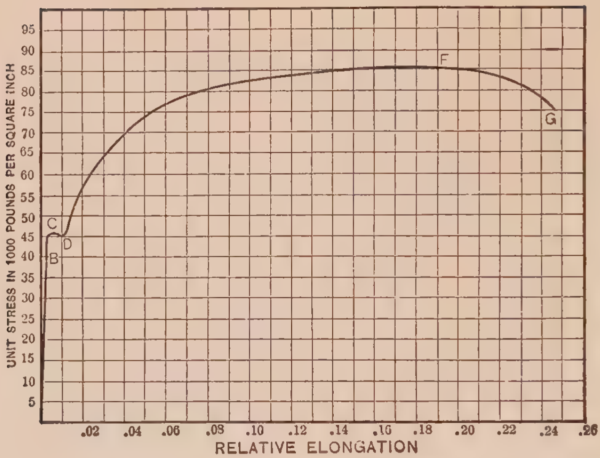


FIG. 7.—Stress-strain diagram of machine steel.

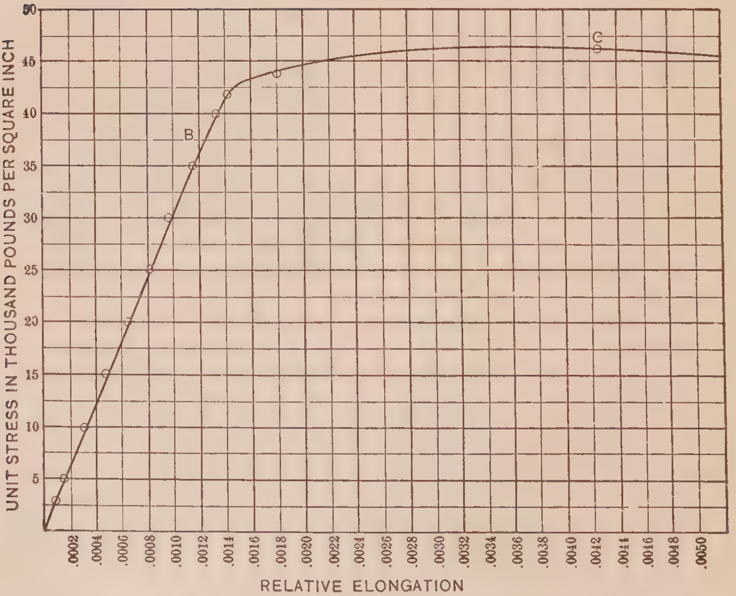


FIG. 8.—Part of diagram for machine steel.

represents a unit stress of 2,500 pounds per square inch (one-half as much as in Fig. 7).

16. Elastic Limit and Yield Point.—In Article 8, *the elastic limit* is defined as the maximum unit stress to which a body may be subjected without permanent deformation. *Elastic limit* is also defined as the unit stress at which the stress-strain diagram begins to deviate from a straight line. Defined in this way, it is called the *proportional elastic limit* or the *proportionality limit.* These two definitions give practically the same unit stress. It is not easy, however, to find this unit stress accurately from either definition. Since the stress-strain diagram deviates at first very gradually from the straight line, the exact point at which the stress ceases to bear a constant ratio to the deformation is often difficult to determine. For stresses near the elastic limit, there are frequently small deformations, which vanish slowly after the removal of the external force. On account of these temporary strains, the determination of the stress at which there is a permanent set is rendered somewhat uncertain.

The point B, of Figs. 7 and 8, is the proportional elastic limit. From Table III, for unit stresses below 35,000 pounds per square inch, the increase in elongation in the gage length is seen to be about 13 divisions for each increment of 5,000 pounds per square inch. Between 35,000 and 40,000 pounds per square inch, the elongation in the gage length is 15.5 divisions. From 40,000 to 42,000 pounds per square inch, the stretch is 6.5 divisions, which is equivalent to 16.2 divisions for a 5,000-pound increase of unit stress. Between 42,000 and 44,000 pounds per square inch, the elongation is 30 divisions, and the rate of increase is more than five times what it is below the proportional elastic limit. From these figures it is evident that the increase in deformation between 35,000 and 40,000 pounds per square inch is not due to an accidental error in measurement, but that there is a definite change in the rate of deformation at some stress slightly above 35,000 pounds per square inch.

At C, at a unit stress of 46,000 pounds per square inch, the curve becomes horizontal. This is the *yield point.* Beyond the yield point the curve drops to a unit stress of about 45,000 pounds per square inch. Not only is there an increase of length with no increase of stress, but there is a considerable elongation with a diminished stress. In changing down to 45,000 pounds and back again to 45,500 pounds, the increase in length is nearly twice

as great as the entire elongation up to the yield point, and five times as great as the elongation from zero load to a stress of 42,000 pounds per square inch.

The unit stress at the yield point may be determined in rapid commercial tests without the use of delicate apparatus for measuring the elongation. Table III shows that the total elongation in a gage length of 8 inches is about $\frac{1}{70}$ inch at the unit stress of 44,000 pounds per square inch and is more than $\frac{1}{30}$ inch at the unit stress of 46,000 pounds per square inch. Since this increase of length may easily be measured by an ordinary scale, the yield point may be determined to within 1,000 or 2,000 pounds per square inch without the use of an extensometer. Again, just beyond the yield point the elongation increases while the load diminishes. This point may be determined in rapid commercial tests in which the testing machine is kept running continuously. Before the yield point is reached, the poise on the beam of the weighing apparatus must be continually moved out to preserve the balance. At the yield point the "beam drops" while the elongation continues to increase. In order to balance the beam, the poise must now be moved backward to a smaller load. Specifications frequently state "The yield point shall be determined by the drop of the beam of the testing machine."* Iron or steel which have not been machined or polished is usually covered with a coat of oxide. When the yield point is reached, flakes of this oxide fall from the test piece. Sometimes a portion of the bar reaches the yield point before the remainder. The oxide breaks loose from this portion while it remains unchanged over the rest of the bar until the load becomes slightly greater. The stress-strain diagram of a rod which yields in this way shows steps which correspond with the yield points of the different portions.

Since the yield point may be determined so easily by methods which were in use before delicate extensometers were available, the term *elastic limit* is sometimes applied to what is really the *yield point*. This confusion of terms is occasionally found in specifications. The present tendency is to employ the term elastic limit to mean the proportionality limit. The late Prof. J. B. Johnson suggested the expression *true elastic limit*, but this term has not come into general use.

* American Society for Testing Materials, Standards, 1921, pages 77, 83, 89, 95, etc.

TABLE IV.—SPECIFICATIONS FOR ROLLED STEEL ADOPTED BY AMERICAN SOCIETY FOR TESTING MATERALS, STANDARDS, 1921

Material		Tensile strength, lb. per sq. in.	Yield point, lb. per sq. in.	Elong. in 8 in., per cent.	Elong. in 2 in., per cent.	Reduction of area, per cent.
Steel for bridges.	Structural	55,000 to 65,000	0.5 tens. str.	$\frac{1{,}500{,}000}{\text{tens. str.}}$	22	
	Rivet	46,000 to 56,000	0.5 tens. str.	$\frac{1{,}500{,}000}{\text{tens. str.}}$		
Structural nickel steel.	Rivet	70,000 to 80,000	45,000	$\frac{1{,}500{,}000}{\text{tens. str.}}$	...	40
	Plates, shapes and bars	85,000 to 100,000	50,000	$\frac{1{,}500{,}000}{\text{tens. str.}}$	...	25
	Eye-bars and pins, annealed	90,000 to 105,000	52,000	20	20	35
Rail-steel for concrete reinforcement bars.	Plain bars	80,000 minimum	50,000	$\frac{1{,}200{,}000}{\text{tens. str.}}$		
	Deformed and hot-twisted bars	80,000 minimum	50,000	$\frac{1{,}000{,}000}{\text{tens. str.}}$		

17. Johnson's Apparent Elastic Limit.—Since it is somewhat difficult to determine the proportional elastic limit accurately, especially in hard steel, where there is a wide range between this stress and the yield point, and in materials (such as cast iron) which have no yield point, the late Prof. J. B. Johnson proposed another point which he called the "*apparent elastic limit.*"* He defined the apparent elastic limit as "the point on the stress diagram at which the rate of deformation is 50 per cent. greater than at the origin." It is that point on the curve at which the slope of the tangent from the *vertical* is 50 per cent. greater than that of the straight-line part of the curve.

This term has not yet come into general use among engineers. In some investigations of the strength of materials, it has been found useful in comparing the results of different tests.†

* See JOHNSON'S "Materials of Construction," pages 18–20.

† See work of H. F. MOORE in *Bulletin* No. 42 and ALBERT J. BECKER in *Bulletin* No. 85 of the University of Illinois Engineering Experiment Station.

18. Calculation of the Modulus of Elasticity.—The stress-strain diagram, when plotted to a sufficiently large scale, makes it easy to calculate the *average* value of the modulus of elasticity. If the straight line passes through the origin, the unit stress is read which corresponds with some convenient unit deformation, such as 0.001 or 0.0005. If the straight-line does not pass through the origin, the difference of unit stress is taken for some convenient difference of deformation. The modulus is then calculated by dividing the difference of unit stress by the difference of deformation.

Problems

1. From the curve of Fig. 8 find the unit stress which corresponds with the unit elongation of 0.0008 and compute E to three significant figures.

2. From Fig. 8 find the unit elongation which accompanies the unit stress of 25,000 pounds per square inch and calculate the modulus of elasticity to three significant figures.

3. From the data of Table III plot the stress-strain diagram up to the unit stress of 42,000 pounds per square inch to the scales 1 inch equals a unit stress of 5,000 pounds per square inch and a unit elongation of 0.0002 inch per inch of length. Use paper ruled in 0.1-inch units. Draw the curve as a light line and solve Problems 1 and 2.

4. From Table III calculate E, using intervals of 15,000 pounds per square inch.

The stress-strain diagram affords a convenient means of finding a fair *average* value of the modulus of elasticity by a single calculation. The diagram also makes it possible to judge of the accuracy of the measurements by observing how nearly the points approach the straight line.

If the modulus of elasticity is computed directly from the readings, and considerable accuracy is desired, it is best to use the average of several values taken with equal intervals of unit stress. Since the errors in reading the extensometer and setting the scale beam are practically constant, the *relative* errors are inversely proportional to the length of the intervals of stress and deformation; for this reason, the intervals should be taken as large as convenient. Four intervals of 20,000 pounds per square inch may be obtained from Table III. These are:

Interval	Elongation in gage length in inches	Modulus in pounds per square inch
0–20,000	0.00535	29,900,000
5,000–25,000	0.00520	30,800,000
10,000–30,000	0.00535	29,900,000
15,000–35,000	0.00510	31,400,000
	Average E	30,500,000

It is not necessary to divide out for the unit elongation. The work may be indicated.

$$20{,}000 \div \frac{0.00535}{8} = \frac{20{,}000 \times 8}{0.00535} = 29{,}900{,}000.$$

On the other hand, it is best to divide the total stress by the area (at least for the largest reading used) since it is desirable to know for what range of unit stress a given modulus of elasticity holds.

19. Ultimate Strength and Breaking Strength.—The point F at the top of the curve of Fig. 7, representing a unit stress of a little more than 85,000 pounds per square inch, gives the *ultimate strength* of the steel under test. The rod at this stress was elongated 1.5 inches in the gage length of 8 inches, and the diameter was practically uniform throughout this length. Beyond this elongation, the rod began to "neck;" its diameter decreasing rapidly at *one section*, while the remainder was not changed. When the load had dropped to about 82,000 pounds per square inch, the minimum diameter at the neck was 0.904 inch, while that of most of the gage length was a little over 1 inch. It finally broke at a total load of 72,000 pounds, which, in terms of the original area, corresponds to a unit stress of 73,800 pounds per square inch. This is the *breaking strength*, the point G of Fig. 7.

Most materials, such as wood, cast iron, concrete and hard steel, do not neck; the ultimate strength corresponds with the breaking strength.

20. Percentage of Elongation and Reduction of Area.—In ductile materials, such as wrought iron and steel, the percentage of elongation is an important factor. In the tested bar of Table III and Fig. 7, the final elongation in 8 inches was 24.7 per cent. The greatest relative elongation is at the neck. To show the variation in elongation, the gage length was subdivided by punch marks into 1-inch spaces. After rupture, the two pieces were

placed together as shown in Fig. 6, II, and these spaces were measured with the following results:

Interval	Elongation
0–1	0.17 inch
1–2	0.19 inch
2–3	0.31 inch
3–4	0.54 inch; included neck
4–5	0.25 inch
5–6	0.19 inch
6–7	0.17 inch
7–8	0.17 inch

If the elongation is taken from the single inch interval 3–4, which included the neck, the result is 54 per cent. From the 4-inch interval, 0–4, the elongation is 30.2 per cent. From the other 4-inch interval, 4–8, the elongation is only 19.5 per cent. In order to make the results of different tests comparable with each other, the Society for Testing Materials has adopted 8 inches as the standard gage length of test bars from most rolled stock. Figure 9 shows the dimensions of a standard test bar of this length as made from a plate.

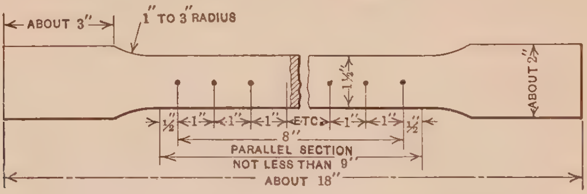


FIG. 9.—Tension test bar—8-inch gage length.

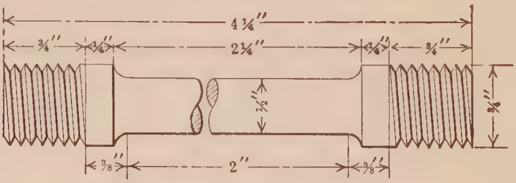


FIG. 10.—Tension test bar—2-inch gage length.

From forgings, castings, and other material from which it is not convenient to take a test bar 18 inches in length, pieces of 2-inch gage length are made, as shown in Fig. 10. The specifications of Table IV sometimes give the percentage of elongation for both the 8-inch and the 2-inch gage length.

The percentage of reduction of area at the neck is also important. In the rod of Table III, the original diameter was 1.115 inches and the final diameter at the neck was 0.821 inch. The final area of the neck 54.2 per cent. of the original area and the reduction of area was 45.8 per cent.

Problems

1. From the above measurements, find the percentage of elongation for the interval 2–5 and also for the interval 1–7.
Ans. 37 per cent., 27.5 per cent.

2. A rod of soft steel, originally 0.874 inch in diameter, was tested in tension. After a fracture under a final load of 23,400 pounds, the gage length of 8 inches was found to be 10.98 inches and the diameter at the neck was 0.499 inch. Find the percentage of elongation and the reduction of area.
Ans. 37.2 per cent. elongation.
67.4 per cent. reduction of area.

3. The maximum load on the test bar of Problem 2 was 32,850 pounds. Find the ultimate tensile strength.
Ans. 54,750 pounds per square inch.

4. Does the steel of Problem 2 meet the specification for rivet steel for bridges as given in Table IV?

5. From Table IV, what should be the relative elongation of structural steel for bridges if the ultimate strength is 62,500 pounds per square inch?
Ans. 24 per cent.

6. What should be the relative elongation of rail-steel deformed bars for concrete reinforcement to satisfy the minimum requirements of the specifications of the Society for Testing Materials, if the ultimate strength is 85,000 pounds per square inch?
Ans. 11.8 per cent.

21. Apparent and Actual Unit Stress.—The unit stresses of Table III were calculated by dividing the total load by the area of the cross-section at the beginning of the test. This is the usual custom and stresses are always so understood unless otherwise designated. Owing to the permanent reduction of area in a ductile material after passing the yield point, the *actual unit stress*, which is calculated by dividing the total load by the actual area of cross-section when loaded, may be much larger. In the bar of Table III, the actual diameter, when the load was 74,176 pounds, was 1.083 inches, and the actual area of cross-section was 0.921 square inches. The actual unit stress was the quotient of 74,176 divided by 0.921 which is 80,540 pounds per square inch, while the apparent unit stress was only 76,000 pounds per square inch.

Problems

1. From Table III calculate the actual unit stress at the neck for the last two loads.
2. From Table III calculate the actual unit stress when the apparent unit stress was 66,000 pounds per square inch.

Before necking begins the actual unit stress may be calculated from the apparent unit stress and the relative elongation. The volume of the gage length remains nearly constant. If A represents the original area of cross-section, the volume of a portion 1 inch in length is equal to A cubic inches. If A' is the area of cross-section when the original inch length is stretched to a length $1 + \delta$, the volume is $A'(1 + \delta)$.

$$A = A'(1 + \delta), \quad A' = \frac{A}{1 + \delta}.$$

$$\text{Actual unit stress} = \frac{P}{A'} = \frac{P}{A}(1 + \delta).$$

Actual unit stress = apparent unit stress multiplied by $(1 + \delta)$.

Problems

3. From Table III, calculate the actual unit stress when the apparent unit stress was 76,000 pounds per square inch. Solve by means of the unit elongations and check by the measured diameter.
4. When the apparent unit stress in Table III was 66,000 pounds per square inch, calculate the diameter by means of the unit elongation and compare with the measured value.

Figure 11 shows the actual and apparent unit stress diagrams for a rod of soft steel. While the curve of apparent unit stress drops when the rod begins to neck, the curve of actual unit stress rises at all points except at the yield point.

An apparent discrepancy may be noticed between the statements of this article and those of Article 14. Poisson's ratio and the theory of Article 14 apply only to the *temporary deformations* inside the elastic limit, while the statement that the volume remains constant applies to the *permanent deformations* beyond the yield point. While the bar is under load, there is also the elastic deformation superimposed on the permanent deformation. The permanent deformation is much greater than the temporary deformation. In so far as the *permanent* deformation is concerned, the material beyond the yield point behaves as if Poisson's ratio were one-half.

22. Curves of Various Structural Materials.—The curves of Figs. 7 and 8 give a fair average idea of the behavior of machine steel in tension. The apparent stress curve of Fig. 11 shows the same for a sample of rather soft steel. This sample was not analyzed but it probably contained less than 0.15 per cent.

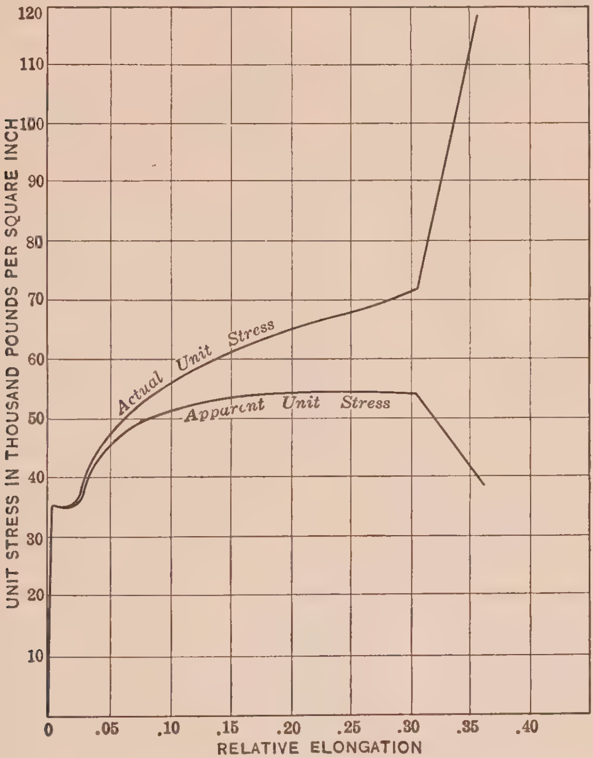


FIG. 11.—Actual and apparent unit stress.

carbon. Its tensile strength was a little below the minimum requirement of structural steel for bridges.

The modulus of elasticity of structural steel is about 29,000,000 pounds per square inch and the yield point a little over one-half the tensile strength.

Tool steel, with 1 per cent. or more of carbon, has a tensile strength of over 100,000 pounds per square inch and a modulus of elasticity of 30,000,000 pounds per square inch. Heat treatment and mechanical treatment may greatly change the ultimate

strength and the yield point of steel, but have little effect upon the modulus of elasticity.

Table V and Curve II of Fig. 12 represent the behavior of cast iron in tension. The table is the mean of the tests of six bars* from the same heat. The figures show what may be expected from good cast iron.

Table V.—Tension Test of Cast Iron

Diameter, 1.129 inches; area, 1 square inch; gage length, 10 inches.

Load per square inch	Elongation	
	In gage length	Per inch length
Pounds	Inch	Inch
1,000	0.00056	0.000056
2,000	0.00112	0.000112
3,000	0.00171	0.000171
4,000	0.00236	0.000236
5,000	0.00303	0.000303
6,000	0.00374	0.000374
7,000	0.00446	0.000446
8,000	0.00526	0.000526
9,000	0.00606	0.000606
10,000	0.00691	0.000691
11,000	0.00779	0.000779
12,000	0.00871	0.000871
13,000	0.00968	0.000968
14,000	0.01061	0.001061
15,000	0.01174	0.001174
16,000	0.01283	0.001283
17,000	0.01404	0.001404
18,000	0.01544	0.001544
19,000	0.01689	0.001689
20,000	0.01851	0.001851
21,000	0.02003	0.002003
22,000	0.02182	0.002182
23,000	0.02420	0.002420
24,000	0.02626	0.002626

* The six bars tested were specimens 8014, 8041, 8050, 8051, 8053, and 8063, Watertown Arsenal, 1905.

The average ultimate load was 26,450 pounds per square inch. The actual initial load was 1,000 pounds. The table is calculated on the assumption that the elongation from 0 to 1,000 is the same as from 1,000 to 2,000.

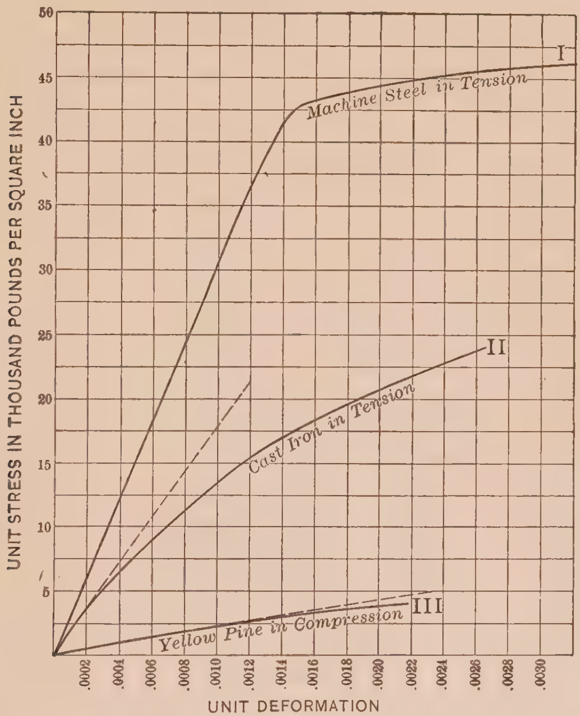


FIG. 12.—Stress-strain diagrams of steel, timber, and cast iron.

The stress-strain diagram for this cast iron was plotted in Fig. 12 to the same scale as was used in Fig. 8, and part of the diagram of steel from Fig. 8 has been drawn for comparison. The broken line, which is tangent to the lower end of the diagram, shows the slope at the initial loads. The curve begins to bend almost from the start, and the elastic limit is difficult to locate. There is no yield point, and the material breaks suddenly without necking.

Problems

1. From the slope of the broken line of Curve II, Fig. 12, calculate the modulus of elasticity of this cast iron. Check by means of the readings of Table V.

TABLE VI.—COMPRESSION TEST OF LONG-LEAF YELLOW PINE

From Watertown Arsenal Report, 1897, page 420.
Length of post, 10 feet. Dimensions, 9.75 inches by 9.77 inches.
Area, 95.26 square inches. Gage length, 50 inches.

Applied load		Deformation	
Total	Unit stress per square inch	In gage length	Unit per inch length
Pounds	Pounds	Inch	Inch
9,526	100	0.0021	0.000042
19,052	200	0.0044	0.000088
28,578	300	0.0067	0.000134
38,104	400	0.0091	0.000182
47,630	500	0.0116	0.000232
57,156	600	0.0141	0.000282
66,682	700	0.0165	0.000330
76,208	800	0.0191	0.000382
85,734	900	0.0215	0.000430
95,260	1,000	0.0240	0.000480
114,312	1,200	0.0290	0.000580
133,364	1,400	0.0340	0.000680
152,416	1,600	0.0389	0.000778
171,468	1,800	0.0443	0.000886
190,520	2,000	0.0495	0.000990
209,572	2,200	0.0546	0.001092
228,624	2,400	0.0601	0.001202
247,676	2,600	0.0652	0.001304
266,728	2,800	0.0705	0.001410
285,780	3,000	0.0758	0.001516
304,832	3,200	0.0811	0.001622
323,884	3,400	0.0869	0.001738
342,936	3,600	0.0932	0.001864
361,988	3,800	0.1005	0.002010
381,040	4,000	0.1077	0.002154
400,092	4,200	0.1084	0.002168
416,000	4,367	*Ultimate strength*	

Failed by crushing at end.

2. From Table V find Johnson's apparent elastic limit for cast iron. Find the difference of elongation for each successive 1,000 pounds, and locate the unit stress at which this difference is 50 per cent. greater than at the beginning.

Table VI and Curve III of Fig. 12 represent the behavior of long-leaf yellow pine in compression. Like that of steel, the stress-strain diagram for timber is a straight line up to a relatively large stress. There is no yield point and the elastic limit is poorly defined. The post represented by Table VI failed outside the gage length. The ultimate deformation was, therefore, less than it would have been if the failure had occurred inside of this length.

Problems

3. Plot Table VI to the scale of 1 inch equals 1,000 pounds per square inch and 1 inch equals 0.0005 relative deformation. Find E and the proportional elastic limit.

4. Find E and the elastic limit of yellow pine from the figures of Table VI, without using the stress-strain diagram.

5. A spruce stick tested at the Bureau of Standards was 1.74 inches square and 12⅝ inches long. When the load changed from 606 pounds to 12,726 pounds, the compression in 8 inches was 0.0173 inch. Find E.

6. The ultimate load on the stick of Problem 5 was 17,600 pounds. Calculate the ultimate strength in pounds per square inch.

7. A 6-inch by 6-inch post, 5 feet long is made of yellow pine and is subjected to a vertical load of 72,000. If the material is the same as that of Table VI how much is the post shortened? What is the total work done by the load? If Poisson's ratio is 0.3 what is the increase in breadth and what is the increase in the area of cross-section?

8. A 6-inch by 6-inch block of long-leaf yellow pine is shortened 0.03 inch in a length of 40 inches by a load parallel to its length. The work done is 648 inch-pounds. Find E and the load.

TABLE VII.—COMPRESSION TEST OF 1:2½:6 CONCRETE; AGE, 90 DAYS

Diameter of test cylinder, 8 inches; area, 50 square inches. Total length, 16 inches; gage length, 10 inches.

Applied load		Deformation	
Total	Per square inch	In gage length	Per inch length
Pounds	Pounds	Inch	Inch
2,000	40	0.00013	0.000013
4,000	80	0.00026	0.000026
6,000	120	0.00038	0.000038
8,000	160	0.00052	0.000052
10,000	200	0.00068	0.000068
12,000	240	0.00081	0.000081
14,000	280	0.00099	0.000099
16,000	320	0.00113	0.000113
18,000	360	0.00136	0.000136
20,000	400	0.00158	0.000158
22,000	440	0.00180	0.000180
24,000	480	0.00206	0.000206
26,000	520	0.00232	0.000232
28,000	560	0.00260	0.000260
30,000	600	0.00295	0.000295
32,000	640	0.00327	0.000327
34,000	680	0.00377	0.000377
36,000	720	0.00421	0.000421
38,000	760	0.00473	0.000473
40,000	800	0.00535	0.000535
42,000	840	0.00609	0.000609
44,000	880	0.00692	0.000692
46,000	920	0.00796	0.000796
48,000	960	0.00922	0.000922
50,000	1,000	0.01058	0.001058
52,000	1,040	0.01177	0.001177
54,000	1,080	0.01323	0.001323
56,000	1,120	0.01575	0.001575
58,000	1,160	0.01847	0.001847
60,000	1,200	*Failed*	

Table VII gives the results of a compression test of a concrete cylinder of rather inferior quality. Figure 13 presents a comparison of this concrete and the yellow pine of Table VI. Curve I of this figure is the stress-strain diagram for the yellow pine.

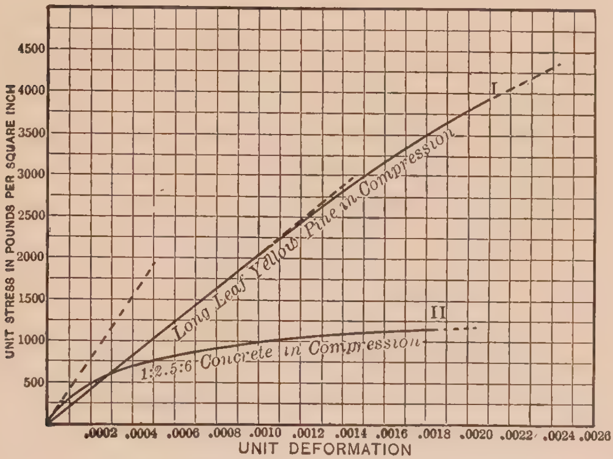


FIG. 13.—Stress-strain diagrams of concrete and yellow pine.

The unit deformations are represented on a scale twice as great as in Fig. 12, and the unit stresses are represented on a scale ten times as great. Curve II is the diagram for the concrete.

Problem

Find the modulus of elasticity of this concrete from Table VII.

Table VIII gives specifications for a number of metals, which are largely used in engineering work. This table and Table IV for rolled steel represent some of the many specifications which have been carefully prepared by Committees of the American Society for Testing Materials, and later have been adopted by the Society. Each specification is based on numerous tests and wide experience in the manufacture and use of the material to which it applies.

TABLE VIII.—SPECIFICATIONS FOR SOME METALS ADOPTED BY THE AMERICAN SOCIETY FOR TESTING MATERIALS, STANDARDS, 1921

Material		Yield point, lb. per sq. in.	Tensile strength, lb. per sq. in.	Gage length	Elongation, per cent.	Reduction of area, per cent.
Class B steel castings	Hard	0.45 tensile strength	80,000	2	15.0	20
	Medium		70,000	2	18.0	25
	Soft		60,000	2	22.0	30
Wrought-iron forgings		0.5 tensile strength	45,000	4	22.0	30
Refined wrought-iron bars		25,000	48,000	8	22.0	
Naval brass rods for structural purposes	Diameter* 1 inch or less	31,000	62,000	2	25.0	
	Over 1 inch to 2.5 inch	30,000	60,000	2	30.0	
	Over 2.5 inch to 3.5 inch	25,000	56,000	2	35.0	
	Over 3.5 inch	22,000	54,000	2	40.0	
Copper bars for locomotive staybolts	Arsenical	......	31,000	2	35.0	
	Non-arsenical	......	30,000	2	30.0	
Medium hard-drawn copper wire	Diameter, in. 0.460		42,000 to 49,000	10	3.75	
	0.229		48,000 to 55,000	10	2.25	
	0.114		50,000 to 57,000	60	1.06	
	0.057		52,000 to 59,000	60	0.94	
	0.040		53,000 to 60,000	60	0.88	
Hard-drawn copper wire	0.460		49,000	10	3.75	
	0.229		59,000	10	1.79	
	0.114		64,300	60	1.02	
	0.057		66,400	60	0.89	
	0.040		67,000	60	0.85	
Soft copper wire	0.460 to 0.290		36,000	10	35.0	
	0.289 to 0.103		37,000	10	30.0	
	0.102 to 0.021		38,500	10	25.0	
	0.020 to 0.003		40,000	10	20.0	
Light aluminum castings alloys	7 to 8.5 per cent. copper		18,000	2	1.0	
	2.5 to 3 per cent. copper, 12.5 to 14.5 per cent. zinc		25,000	2	1.0	
	2 to 2.5 per cent. copper, 0.75 to 1.25 per cent. manganese		18,000	2	8.0	
Malleable-iron castings			45,000	2	7 5	
Gray-iron castings	Light castings		18,000			
	Medium castings		21,000			
	Heavy castings		24,000			

* Diameter here means the diameter or thickness between parallel faces of the bar from which the test piece is taken.

23. Factor of Safety.—In Article 6, the allowable unit stress was said to be based on the judgment of some competent authority. These judgments depend on tests of materials, such as those of Tables III, V, VI and VII, and on experience in actual use.

Working stresses should never exceed the elastic limit, and should be only a small fraction of the ultimate strength. The ratio of the ultimate strength of a material to the allowable working stress is called the factor of safety. If the ultimate tensile strength of a given grade of steel is 64,000 pounds per square inch and the elastic limit is 32,000 pounds per square inch, while the allowable stress is 16,000 pounds per square inch, the factor of safety based on the ultimate strength is 4, and based on the elastic limit is 2.

Problems

1. If the steel of Table III is used with a factor of safety of 4, what is the allowable unit stress.

2. With a factor of safety of 5, what should be the allowable unit stress in spruce such as that described in Problem 6 of Article 22?

3. What is the safe load with a factor of 4 on a hard drawn copper wire 0.229 inch in diameter, which meets the minimum requirements of the A. S. T. M. Specifications?

4. Cast iron in tension is used with the allowable unit stress given in Table I. What is the factor of safety?

The value of the factor of safety depends upon a great number of conditions. Some of these are:

Repeated stresses slightly beyond the elastic limit will finally cause failure; therefore a body subjected to a variable load should have its allowable stresses well below that limit. The greater the variation of stress the smaller should be the allowable unit stress.

The factor of safety must be sufficiently large to allow for any deterioration of the material during the time which it is to be used. This includes the decay of timber, the rusting of metal, injury from frost, and electrolysis.

In deciding what factor of safety to use, the uniformity of the material must be taken into account. Structural steel which has an ultimate strength of 60,000 pounds per square inch on an average will seldom vary 5,000 pounds on either side of this figure; while the variation of timber sufficiently good to pass a reasonable inspection may be 50 per cent. of the average ulti-

mate strength. An engineer, in designing a concrete structure which he knows will be built under competent supervision, will use much higher unit stresses than he will risk where such inspection is wanting.

The factor of safety must depend also upon the damage which would occur if the material should fail. A workman might use a plank with a small factor in a scaffold 3 feet above the ground, but would demand an ample factor if failure meant a fall of 100 feet.

The factor of safety must allow some margin for unexpected and unreasonable loads. That part of the factor of safety which makes allowance for lack of ordinary judgment in the persons using the machine or structure is called the "fool factor."

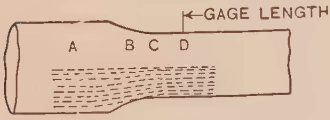


Fig. 14.—Stress distribution in test bar.

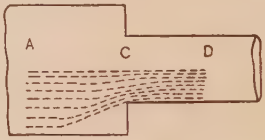


Fig. 15.—Abrupt change of section.

24. Effect of Form on the Ultimate Strength.—In the discussion of stress, it has been assumed that the stress across any section is uniform. This is true in a rod of uniform cross-section at some distance from the surface at which the load is applied, provided the line of the resultant force coincides with the axis of the rod. In order to have the stress evenly distributed and the same at all sections, test bars are made of uniform section throughout a portion somewhat longer than the gage length. In Fig. 9, for instance, the gage length is 8 inches, while the parallel portion is not less than 9 inches. When the gage length is made of smaller sections than the ends of the bar, the change of section is gradual, as is shown in Fig. 14. The broken lines in Figs. 14 and 15 are intended to represent the flow of stress in the bar. At section C, there is a crowding of the lines near the surface. It the gage length extended to C at the beginning of the parallel portion, the measured elongation would be too high, since the stress at the surface is greater than the average of the section. If the change in section were abrupt, as in Fig. 15, the error in elongation would be still greater.

The *ultimate strength* of a rod, at a point at which the section changes, depends upon the ductility of the material. A rod of

cast iron or other *non ductile* material in the form of Fig. 15, will fail at section C, where there is a concentration of stress near the surface. The more abrupt the change of section, the greater is the concentration of stress and the easier the failure.

If the rod is made of *ductile* material, such as structural steel, the strength at C is increased by material of the larger section to the left. A ductile substance necks before it fails. The material of the larger portion tends to prevent necking in the smaller portion for some distance to the right of C. Rods of ductile material with short reduced area, such as I and II

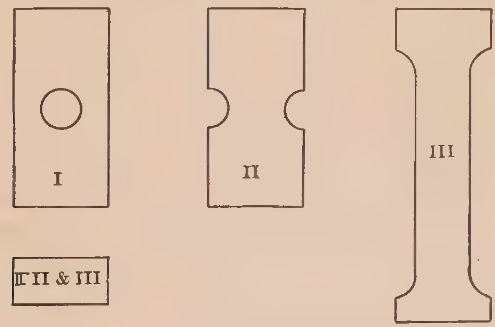


FIG. 16.—Reduced sections.

of Fig. 16 will show a higher ultimate strength than a rod in which the minimum section is longer, as in III, Fig. 16. In rods I and II the portion with the minimum section is close to the larger sections which hinder the necking. In rod III, on the other hand, most of the portion of minimum section is so far removed from the larger sections that necking takes place without hindrance.

It is not necessary to make test bars of the form shown in Figs. 9 and 10; any bar of uniform section will do, and many tests are made of such bars as they come from the rolls. There is this advantage in the standard form shown in Figs. 9 and 10—that it will fail inside the gage length on account of the resistance to necking for some distance from the larger section. A bar of uniform section may fail outside of the gage length.

It is hardly necessary to state that all changes in section should be gradual. The standard form of bar, as adopted by the Society for Testing Materials (Figs. 9 and 10) changes from large to small section on the arc of a circle tangent to the surface of the

smaller section. It is easier to make a *taper* from one size to the other, and the results are practically as good.

25. Effect of Stresses beyond the Yield Point.—In materials which are not ductile, any stress beyond the elastic limit produces a permanent injury. In ductile materials, especially soft steel and iron, this is not the case. If a rod of soft steel, originally hot-rolled, is stressed beyond the yield point, there is some permanent deformation. If the load is partly or entirely released and again applied, it will be found that the yield point has been raised. Suppose a rod of steel which has a yield point of 35,000 pounds per square inch is carried up to 50,000 pounds per square inch. When this rod is again loaded, the yield point will be about 50,000 pounds per square inch. The exact value of the second yield point will depend somewhat upon the speed at which the two loads are applied.

FIG. 17.—Soft steel in tension; left, cold-rolled; right, annealed.

When a high elastic limit and yield point are desired, soft steel is subjected to cold rolling. Figure 17 shows the effect of cold rolling. The middle rod is a piece of $\frac{7}{8}$-inch cold-rolled shafting. The left one is an exactly similar rod after testing in tension. Its ultimate strength was over 86,000 pounds per square inch, its yield point was about 80,000 pounds per square inch, and its elongation about 10 per cent. On the right is a third rod, originally like the others, which was annealed by heating to redness and slowly cooling to destroy the effect of the previous cold rolling. When tested in tension, its ultimate strength was found to be 60,000 pounds per square inch, its yield point was 40,000 pounds per square inch, and its elongation 22 per cent. It will be seen from these tests that cold rolling raises the yield point to nearly the ultimate strength and that it increases the ultimate strength a considerable amount.

The fact that soft steel may be stressed beyond the yield point without injury, and with no change except a slight reduction of section and elevation of the yield point, is of great advantage in its use in structures. In a heavy structure made of many parts, there is always some adjustment when the loads are first applied. This may cause an overstraining of some parts. If these parts are made of soft steel, they can yield slightly, permitting other members to take part of the excess load.

Miscellaneous Problems

1. A bronze bar tested at Watertown Arsenal (1911, page 118) was 88 per cent. copper, 10 per cent. tin, and 2 per cent. zinc. The diameter was 0.798 inch and the gage length was 10 inches. When the total load changed from 500 pounds to 5,500 pounds, the elongation in the gage length was 0.0080 inch. The ultimate load was 15,900 pounds. The elongation in the gage length was 1.04 inch. The diameter at the fracture was 0.77 inch. Find the modulus of elasticity, the ultimate strength, the per cent. of elongation, and the per cent. of reduction of area.

2. A cast steel rod tested at Watertown Arsenal (1911, page 82) had an ultimate strength of 82,500 pounds per square inch, an elastic limit of 45,500 pounds per square inch, an elongation of 15 per cent. in a gage length of 2 inches, and a reduction of area of 24 per cent. Did this steel meet the specifications of the American Society for Testing Materials?

CHAPTER III

SHEAR

26. Shear and Shearing Stress.—When a body is subjected to a pair of forces which are in the *same line* and directed *away* from each other, *tensile* stress is produced. When the pair of forces are in the *same line* and directed *toward* each other, *compressive* stress is produced. If the forces are in *parallel lines* or *planes*, *shearing* and bending stresses are produced in the portion of the body between them. In Fig. 18, the block A is securely held by the clamp B and a horizontal force P is applied by a block C. The force P is parallel to the upper surface of B. The clamp B exerts a horizontal force on the block A. This force is equal and opposite to the force P. The portion of the block A between the upper surface of the clamp and the lower surface EFG of the block C is subjected to a pair of equal, opposite, parallel forces. The material in this portion of the block is subjected to shearing and bending stresses. The shearing stresses depend upon the magnitude of the forces and the area of the section of A. The bending stresses depend upon these and also upon the distance of the forces apart. If the body C is brought very close to B, so that the distance between the two forces P and P' becomes negligible, the unit bending stress becomes small, while the unit shearing stress is unchanged. The average unit shearing stress is calculated by dividing the force P by the area of the cross-section EFG or the area of any section parallel to it.

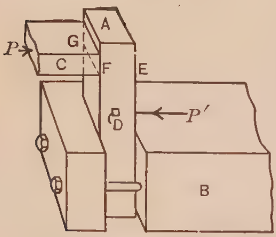


FIG. 18.—Shear and bending.

In tension and compression the unit stress is calculated by dividing the total force by the area of the cross-section perpendicular to it. In shear, on the other hand, the unit stress is calculated by dividing the total force by the area of the cross-section parallel to it.

In Fig. 18, as in all cases of application of force, the line P represents the resultant of a set of forces distributed over an area. The resultant P' must fall some distance below the upper surface of B and the resultant P must lie above the lower surface of C. It is, therefore, not practicable to secure shearing stress entirely free from bending or compressive stress. It will be shown later that the distribution of shearing stress, when combined with bending, is not uniform over the section. At present, however, no account will be taken of this variation, and the average shearing stress will be calculated by dividing the total force by the area in shear.

TABLE IX.—ALLOWABLE UNIT SHEARING STRESS
(To be memorized)

Material	Pounds per square inch
Steel web plates for bridges	10,000
Power driven rivets	12,000
Hand driven rivets	9,000
Steel rivets in boilers	8,800
Long-leaf pine parallel to grain	120

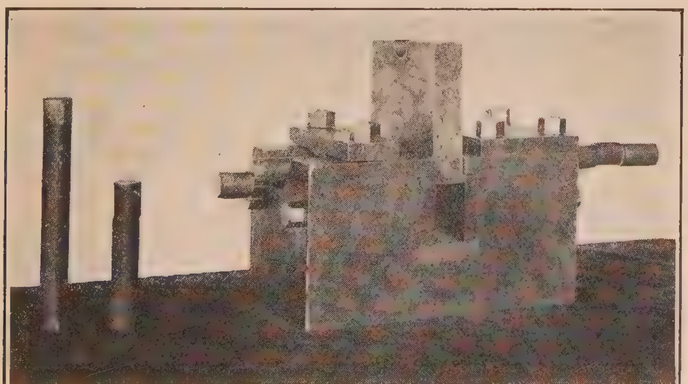


FIG. 19.

Figure 19 shows the apparatus for testing rods for the ultimate shearing strength. The test bar rests on two parallel plates, which are about one inch apart and is pushed down by a movable plate, which passes between the two fixed plates. The apparatus

is arranged for a ¾-inch bar, which rests in ¾-inch hemi-cylindrical grooves at the upper edges of the fixed plates. There is a similar ¾-inch groove at the lower edge of the movable plate where it rests on the bar. (The photograph shows a 1-inch groove at the top of the movable plate. To test a 1-inch rod, another pair of fixed plates, provided with 1-inch grooves are fastened in the large block of steel and the movable plate is turned over.) To reduce the bending, the test bar is held down on each side of the fixed plates by horizontal plates, which are bolted to the large block. To make the test, the large block is placed on the weighing table of a testing machine and the movable plate is pushed down by the cross-head.

The ¾-inch bar in Fig. 19 had previously been tested at two places. The right end of the photograph shows the result of one test (the bar has been turned forward from the position in which it was tested). The deformation is 0.15 inch. While the load at this deformation was a few thousand pounds smaller than the ultimate load, the ductility of the metal is so great that the bar is still firmly held together at each shear plane. At the left end of the photograph, the deformation is 0.30 inch. The rod is still held together at one shear plane, but has failed at the other plane. The longer bar standing at the left of the photograph is the remainder of the test piece.

The test bar shown on the apparatus of Fig. 19 is cold-rolled steel which has been annealed to remove the effects of cold rolling. It is equivalent, therefore, to soft, hot-rolled steel. The shorter bar standing at the left is a piece of the same cold-rolled steel, which has not been annealed. This bar failed in shear at one surface when the total deformation was only 0.04 inch. The ultimate shearing strength, however, was greater than that of the softer bar.

While the test bar is held firmly in this apparatus, and the plates by which the load is applied are close together, there is still some bending. The shear failure is partly tension failure. The effect of tension beyond the yield point is shown at the top of the longer standing bar. For a short distance from the top, which was the shear plane (at the left end of the horizontal bar) the scale has broken off. The same effect is shown on the horizontal bar at the right of the second shear plane.

Problems

1. The soft steel rod of Fig. 19 was ¾ inch in diameter. It failed in shear under a load of 30,200 pounds. Find the unit shearing strength of this steel.

   Ans. 34,180 pounds per square inch.
2. The cold-rolled rod of Fig. 19 was ¾ inch in diameter and failed under a load of 38,000 pounds. Find the ultimate shearing stress.

   Ans. 43,000 pounds per square inch.
3. A 4-inch by 4-inch block has a notch cut in one side. The edge of the notch is 10 inches from the end of the block. A pull of 4,000 pounds is applied by means of a second block set in the notch. What is the unit shearing stress? Is the construction safe if the block is yellow pine?
4. Steel rivets are made of material equivalent to that of Problem 1 and are power driven. What is the factor of safety?
5. A 2-inch by 4-inch yellow pine block, hung vertical and supported at the upper end, has a hole 1 inch square perpendicular to the 4-inch faces. The lower edge of this hole is 4½ inches above the lower end of this block. If a load of 1,620 pounds is hung on a square bar passing through this hole, what is the shearing stress? (Fig. 20.)
6. The head of a 1-inch bolt is 13/16 inch thick. Find the unit shearing stress tending to strip the head from the bolt when there is a pull of 10,000 pounds.

   Ans. 3,918 pounds per square inch.
7. The load in Problem 6 is applied by means of a nut. The dimensions and threads are Franklin Institute standard. (See Cambria or Carnegie handbook.) Find the unit tensile stress in the net section. Find also the unit shearing stress in the threads if the nut is a perfect fit.

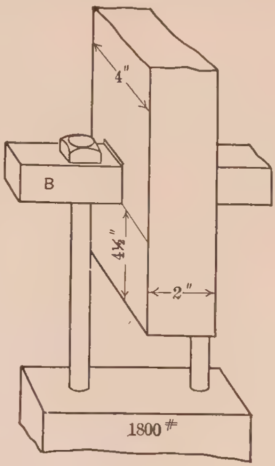


FIG. 20.—Shear in timber.

27. Shearing Deformation.—In Fig. 18, a small portion of section *D* extends through the block *A* with its long dimension perpendicular to the plane which contains the resultants *P* and *P′*. The cross-section *D* is represented on a large scale by the rectangle *HIJK* of Fig. 21. When the shearing forces are applied as shown in Fig. 18, this rectangle is distorted to the form *HI′J′K*. If the lower line *HK* be regarded as fixed, the total displacement of any point in the upper line is *II′* or *JJ′*. The unit shearing deformation, which may be represented by δ_s, is the ratio of this horizontal displacement *II′* to the vertical distance *HI*. In linear deformation, the unit deformation is obtained by dividing the total deformation by a length in

the *same direction as the deformation;* in shearing deformation, the displacement is divided by a distance at *right angles* to the *displacement.* The unit displacement is the tangent of the angle IHI' or JKJ'. The effect of the shearing forces is to lengthen the diagonal HJ, and shorten the diagonal IK.

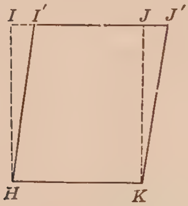


FIG. 21.—Shearing deformations.

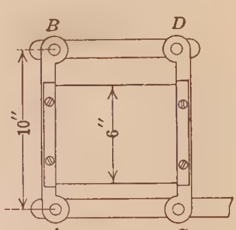


FIG. 22.—Device for illustrating shear.

Problems

1. Two equal bars, AB and CD, Fig. 22, are hinged to a second pair of equal bars, AC and BD, to form a parallelogram. A sheet of rubber, 6 inches wide, has one edge securely clamped to AB and the other edge to CD. The length of AB, center to center of hinges, is 10 inches. What is the unit shearing displacement when B is displaced 0.2 inch to the right of the vertical? *Ans.* Unit shear, $\delta_s = 0.02$.
2. A hollow circular shaft, 5 inches in diameter, is subjected to a twisting moment, and it is found that two sections, 10 feet apart, suffer a relative displacement of 2 degrees. What is the total shearing displacement of the fibers? What is the unit displacement?
Ans. Total displacement, 0.0873 inch.
Unit displacement, 0.0007275.

28. Modulus of Elasticity in Shear.—The modulus of elasticity in shear is obtained by dividing the unit shearing stress by the unit shearing deformation, just as the modulus of elasticity in tension or compression is computed by dividing the unit tensile or compressive stress by the corresponding unit deformation.

$$E_s = \frac{S_s}{\delta_s}.$$

The modulus of shearing elasticity is frequently called the modulus of rigidity.

Forces applied as in Fig. 18 do not give pure shear. Even in Fig. 19, in which the plates which apply the parallel forces are as close together as possible, shear is combined with bending. Pure shear, free from bending or compression may be secured by torsion, as in Problem 2 of Article 27.

Problems

1. In Problem 2 of Article 27, if E_s is 11,600,000 pounds per square inch, what is the unit shearing stress? *Ans.* 8,440 pounds per square inch.
2. What is the maximum allowable unit shearing deformation, if the maximum allowable unit shearing stress is 10,000 pounds per square inch and the modulus of rigidity is 11,200,000 pounds per square inch? *Ans.* 0.000893.
3. A 4-inch shaft, 10 feet long, is subjected to a twisting moment. One end is fixed. How much may the other end move if the unit stress does not exceed 8,000 pounds per square inch in the outer fibers and E_s = 11,000,000 per square inch? What will be the angle of twist in radians, and what will it be in degrees?

29. Shear Caused by Compression or Tension.—Figure 23 shows a block subjected to a downward compressive force P in the direction of its length and an equal upward force at the

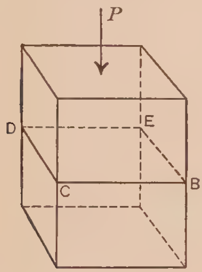


FIG. 23.—Section normal to force.

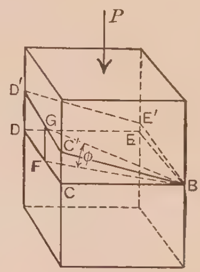


FIG. 24.—Section inclined to force.

bottom. This block may be supposed to be cut by a plane $BCDE$, normal to its length, and then glued together. If the portion above this section be regarded as a free body in equilibrium, and if the weight of the portion be neglected, the downward force P must be equal to the upward reaction of the glued surface. If A is the area of the section, the unit compressive stress in the glue is given by

$$s_c = \frac{P}{A}, \qquad (1)$$

Since the external force P has no horizontal component, the shearing force in the glue is zero. If the body were actually made of two portions, the upper portion would not slide on the lower portion, no matter how smooth the surface of contact.

Figure 24 represents a body similar to Fig. 23, and loaded and supported in the same way. This body is assumed to be cut by a plane $BC'D'E'$, which makes an angle ϕ with the normal section.

The portion above the inclined section may be taken as the free body, and the external force P may be resolved perpendicular and parallel to this plane. The component of P normal to the plane is $P \cos \phi$. The unit compressive stress is this component divided by the area of the section. If A is the area of the normal section, the area of the inclined section is $A \sec \phi$. The unit compressive stress is given by

$$s_c = \frac{P \cos \phi}{A \sec \phi} = \frac{P}{A} \cos^2 \phi = \frac{P}{2A}(1 + \cos 2\phi). \qquad (2)$$

The component of the force P in the direction of the line BG, which makes the maximum angle with the normal plane, is $P \sin \phi$. This component is resisted by the shearing stress in the section $BC'D'E'$. The unit shearing stress is obtained by dividing the component parallel to the section by the area of the section.

$$s_s = \frac{P \sin \phi}{A \sec \phi} = \frac{P}{A} \sin \phi \cos \phi = \frac{P}{2A} \sin 2\phi. \qquad (3)$$

If the body were in tension instead of compression, Equation (3) would still give the unit shearing stress in the section, and Equation (2) would give the unit tensile stress (instead of the unit compressive stress) normal to the section.

Problems

1. A 6-inch by 4-inch post is cut by a plane which makes an angle of 25 degrees with the 4-inch faces and is normal to the 6-inch faces. What is the length of the intersection of this plane with the 6-inch faces? What is the area of the section? If a load of 7,200 pounds is placed on this post, what is its component down the inclined plane, and what is the component normal to the plane? What is the unit shearing stress along this inclined plane? What is the unit compressive stress normal to the plane? Solve without using Equations (2) or (3).

2. Solve Problem 1 for the unit compressive and unit shearing stress by means of Equations (2) and (3).

3. Show from Equations (2) and (3) that the shearing stress is zero and the compressive stress is a maximum when ϕ is zero.

4. A 6-inch by 6-inch post is subjected to a load of 14,400 pounds in the direction of its length. Find the unit shearing stress and the unit compressive stress with respect to a plane which makes an angle of 20 degrees with the normal section.

5. Solve Problem 4 if the plane makes an angle of 70 degrees with the normal section. *Ans.* $s_c =$ 47 pounds per square inch; $s_s =$ 129 pounds per square inch.

6. What is the area of the section in Problem 4, and what is the area in Problem 5?
7. In a 4-inch by 4-inch yellow pine post, the grain makes an angle of 10 degrees with the direction of the length. Find the allowable load. Solve also if the grain makes an angle of 5 degrees with the direction of the length. *Ans.* 11,230 pounds; 19,200 pounds.
8. Prove that the unit shearing stress produced by a single tensile or compressive load is a maximum at 45 degrees with the direction of the load, and that the maximum unit shearing stress is one-half of the tensile or compressive stress which produces it.

30. Shearing Forces in Pairs.—If a body is subjected to pure shearing stress (with no tension or compression except that which is due to shear), there must be two sets of shearing forces to secure equilibrium and the unit shearing stresses which these

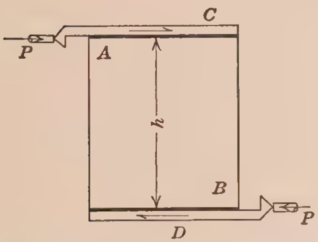


FIG. 25.—Pair of shearing forces.

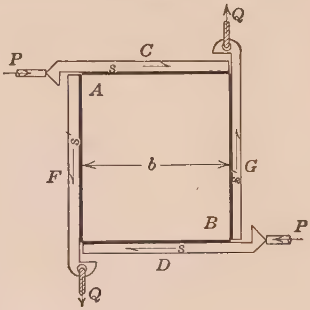


FIG. 26.—Two pairs of shearing forces.

forces produce must be the same in both. Figure 25 represents a rectangular block AB with two other blocks C and D glued to the top and bottom, respectively. There is a horizontal force P, toward the right, acting on the block C and an equal and opposite force acting on the block D. These two forces form a couple tending to rotate the system in a clockwise direction. To produce equilibrium, a block F is glued to the left vertical face of AB (Fig. 26) and a block G is glued to the right vertical face. A downward force Q is applied to F and an equal upward force is applied to G. The breadth of AB is b and its height is h. Equilibrium will occur when the moments of the two couples are equal, that is, when

$$Ph = Qb. \tag{1}$$

The force is transmitted from C and D to AB as a horizontal shear in the glue. Shearing stress is represented by an arrow with a single barb. The arrow in C, with barb above the shaft,

represents the shearing stress from C to AB. If it were desired to represent the opposite shearing stress from AB to C, the arrow would be placed in AB, would point toward the left, and would have the barb below the shaft.

If l is the length of the block AB perpendicular to the plane of the paper, the top and bottom surfaces each have an area bl, and

$$P = sbl, \tag{2}$$

in which s is the unit horizontal shearing stress.

The area of each vertical face perpendicular to the plane of the paper is hl and

$$Q = s'hl,$$

in which s' is the unit vertical shearing stress.

Since
$$Ph = Qb,$$
$$sblh = s'hlb, \tag{4}$$
$$s = s'. \qquad \text{Formula III.}$$

Formula III applies to any portion of block AB cut out by horizontal and vertical planes perpendicular to the plane of the paper. Figure 27 represents one such block.

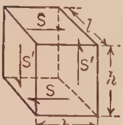


FIG. 27.—Equilibrium in shear.

Frequently, tensile or compressive stresses occur along with shearing stresses. Figure 28 represents a block which is supposed to be glued to the base and pushed toward the right by a force P applied near the top. To the left of the middle, the glue is in tension; to the right of the middle it is in compression. All of the glue is in shear. A portion A of the block is in tension and shear, and a portion C is in compression and shear. The portion B at the middle is in shear only. The direction of the shear in A and C, for which the arrows are not shown, is the same as in B.

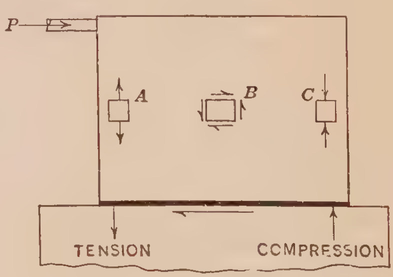


FIG. 28.—Shear with tension and compression.

If the tension in A is not the same at the top and bottom, the vertical shearing stress will not be exactly equal on the two sides. Ordinarily, if A is small, the difference is slight.

For a block of infinitesimal dimensions, the shearing stresses are practically equal on all sides, even if there are tensile and compressive stresses. The combination of shear with other stresses is considered at greater length in Chapter X.

31. Compressive and Tensile Stress Caused by Shear.—Figure 29, I, represents a rectangular parallelopiped of breadth b, height h, and length l, subjected to *pure* shearing stress. The shearing stress acts toward the right parallel to the breadth at the top and toward the left at the bottom. As shown in Article 30, there is also a shearing stress of the same intensity at the left surface acting downward and an equal shearing stress at the right surface acting upward.

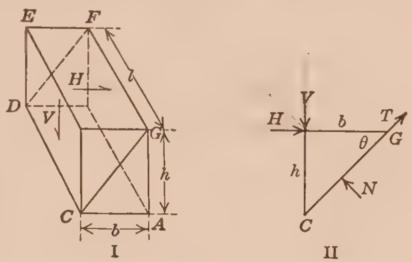


FIG. 29.—Shear causing compression.

(If the direction of one of these shears is reversed, they must all be reversed to produce equilibrium.) Now consider the parallelopiped divided by the inclined plane containing the edges CD and GF, and treat the triangular prism to the left of this plane as a free body in equilibrium under the action of the forces at its surface. These forces are four in number: the shearing force H in the upper surface acting toward the right, the shearing force V in the left vertical surface acting downward, the compressive force N acting normal to the inclined surface (Fig. 29, II, which represents all the forces in the plane of the paper), and a shearing force T along this surface parallel to the diagonal line CG. If s_s is the intensity of the horizontal and vertical shear,

$$H = s_s bl, \quad V = s_s hl.$$

Resolving normal to the inclined plane,

$$N = H \sin \theta + V \cos \theta, \tag{1}$$

$$N = s_s bl \sin \theta + s_s hl \cos \theta, \tag{2}$$

in which θ is the angle which the inclined plane makes with the horizontal surface. The unit compressive stress on the inclined surface is obtained by dividing N by the area of this

surface. If c is the length of the hypotenuse CG, the area of the inclined surface is cl. Dividing Equation (2),

$$s_c = \frac{s_s b \sin \theta}{c} + \frac{s_s h \cos \theta}{c}. \tag{3}$$

Since $\cos \theta = \frac{b}{c}$ and $\sin \theta = \frac{h}{c}$,

$$s_c = 2s_s \sin \theta \cos \theta = s_s \sin 2\theta. \tag{4}$$

When θ is 45 degrees, the compressive stress is a maximum and is equal to the shearing stress.

$$s_c = s_s. \qquad \text{Formula IV.}$$

If the plane which bisects the parallelopiped be taken parallel to CD through A and E, an equation for the tensile stress is found similar to Equation (4). The maximum tensile stress is at right angles to the maximum compressive stress. When a body is subjected to pure shear, there is a compressive stress of equal intensity at an angle of 45 degrees with the planes of the shearing stresses in one direction and a tensile stress of the same intensity at an angle of 45 degrees in the opposite direction. This is shown in Fig. 30. With the shearing stress toward the left at the bottom, as shown by the arrow, the maximum tensile stress is normal to a plane which makes an angle of 45 degrees to the left of the vertical upward, and the maximum compressive stress is normal to a plane which makes an angle of 45 degrees to the right of the vertical upward.

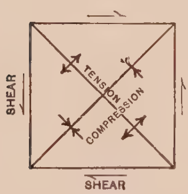


FIG. 30.—Tension, compression and shear.

Problems

1. The unit shearing stress in a block is 120 pounds per square inch, and is directed toward the left at the bottom. Find the unit compressive stress across a plane which makes an angle of 20 degrees with the horizontal. Also find the unit compressive stress across a plane which makes an angle of 70 degrees with the horizontal. *Ans.* 77 pounds per square inch.
2. A block 6 inches long, 4 inches wide, and 5 inches high is glued to a horizontal surface. A horizontal force of 4,800 pounds, directed toward the right, is applied to the left side near the top. Find the unit shearing stress in the block. Find the maximum unit compressive stress caused by shear.

32. Relation of Shearing to Linear Elasticity.—The modulus of shearing elasticity may be calculated from the modulus in tension or compression if Poisson's ratio is known.

Figure 31 is the front elevation of a block of square section subjected to shearing forces. The unit shearing displacement is the tangent of the angle θ between the lines HI and HI' of

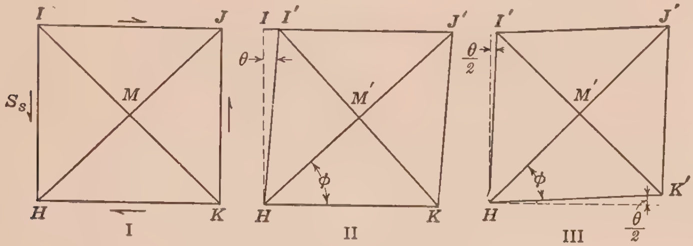


FIG. 31.—Shearing deformation.

Fig. 31, II. In Fig. 31, III, the figure has been rotated an amount equal to one-half the angle of shear. In this position, the diagonals make angles of 45 degrees with the horizontal. The lengths of the diagonals are changed, while their directions remain the same as in Fig. 31, I. This kind of deformation is called *pure shear*. The deformation shown in Fig. 31, II, in which both length and direction of the diagonals are changed, while the direction of one pair of faces is unchanged, is called *simple shear*.

To find the ratio of shearing deformation to linear deformation, it is necessary to find the relation of the diagonals HJ' and $I'K'$ of Fig. 31, III, to the angle $\frac{\theta}{2}$. The shearing forces lengthen the diagonal HJ to HJ' and shorten the diagonal IK to $I'K'$. The half diagonals, HM and MK suffer the same relative deformation.

If δ is the unit deformation caused by unit tensile stress s, the unit elongation along HJ is $\delta(1 + \sigma)$. This elongation is made up of the elongation δ caused by the tensile stress along this diagonal and an elongation $\sigma\ \delta$ caused by the compressive stress along the diagonal IK. In a similar manner, the unit compression along the diagonal IK is found to be $\delta(1 + \sigma)$.

In Fig. 31, III, the angle $\frac{\theta}{2}$ is the difference between 45 degrees and the angle ϕ. The tangent of ϕ is the ratio of the half diagonal $M'K'$ to the half diagonal HM'

$$\tan\phi = \frac{M'K'}{HM'} = \frac{MK[1-\delta(1+\sigma)]}{HM[1+\delta(1+\sigma)]} = \frac{1-\delta(1+\sigma)}{1+\delta(1+\sigma)}. \qquad (1)$$

$$\text{Tan}\,\frac{\theta}{2} = \frac{\tan 45^\circ - \tan\phi}{1+\tan 45^\circ \tan\phi} = \delta(1+\sigma). \qquad (2)$$

For small angles $\tan\theta = 2\tan\dfrac{\theta}{2} = 2\delta(1+\sigma)$. (3)

Since, $\delta = \dfrac{s}{E}$; $\tan\theta = \dfrac{2s(1+\sigma)}{E}$,

$$E_s = \frac{s_s}{\delta_s} = \frac{s_s}{\tan\theta} = \frac{s_sE}{2s(1+\sigma)}. \qquad (4)$$

At 45 degrees the unit tensile and the unit compressive stress caused by shear are each equal to the unit shearing stress: therefore, $s = s_s$ and equation (4) becomes

$$E_s = \frac{E}{2(1+\sigma)}. \qquad (5)$$

Problems

1. If Poisson's ratio is ¼, show that $E_s = \dfrac{2E}{5}$.

2. If the modulus of elasticity of steel is 29,600,000 pounds per square inch and Poisson's ratio is 0.27, what is the modulus of rigidity?

Ans. E_s = 11,650,000 pounds per square inch.

3. Landolt and Börnstein give the following values, in kilograms per square millimeter for cast steel: E = 20,400, E_s = 8,070, E_v = 14,600. Find Poisson's ratio by means of equation (5), and also by means of equation (9), Article 14.

Miscellaneous Problems

1. Figure 32 shows a block which is subjected to horizontal tension and vertical compression. If the unit tensile stress is 200 pounds per square

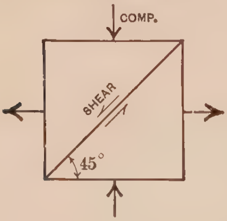


FIG. 32.—Shear caused by tension and compression.

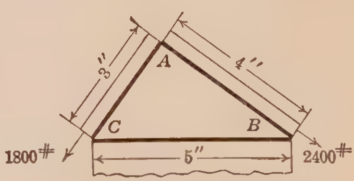


FIG. 33.—Stress due to shear.

inch and the unit compressive stress is 120 pounds per square inch, what is the unit shearing stress at 45 degrees with the horizontal?

Ans. 160 pounds per square inch.

2. If in Fig. 32, the unit compressive stress is equal to the unit tensile stress, show that the unit shearing stress at 45 degrees is the same as each of the others.
3. In Fig. 33, the block ABC is 6 inches long perpendicular to the plane of the paper. Find the unit shearing stress and the unit compressive stress in the glue at the base.

   Ans. s_s, 28 pounds per square inch.
   s_c, 96 pounds per square inch.
4. If the ultimate shearing strength of the plate is 42,000 pounds per square inch, what force is required to punch a 1-inch hole in a ⅝-inch plate?

   Ans. 82,470 pounds.
5. In Problem 4, what is the unit compressive stress in the punch?

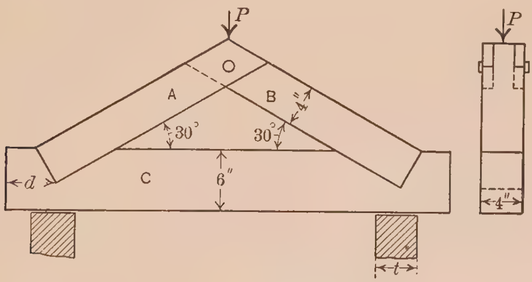


FIG. 34.—Stresses in a truss.

6. In Fig. 34, A and B are short compression members or struts of yellow pine, joined together at the top by a bolt or pin and held from spreading at the bottom by being set into the notches in the bottom chord C. If the load P is 6,000 pounds, what is the unit compressive stress in A and B? What is the maximum unit tensile stress in C? What must be the length of the section d to avoid shearing, if C is made of yellow pine?

   Ans. Length of d, 10.8 inches.
7. A horizontal beam is 5 feet long and is hinged at the left end A. The beam carries a load of 1,200 pounds 3 feet from A and is supported by a steel rod at the right end B. The rod is attached to a point which is 6 feet above the hinge. What should be the minimum diameter of the rod?
8. A beam similar to that of Problem 7 is supported by a rod which is fastened to the right end B and to a point which is directly over the hinge A. What angle should this rod make with the horizontal in order that its weight may be a minimum?

CHAPTER IV

RIVETED JOINTS

33. Kinds of Stress.—Riveted joints afford an excellent illustration of tension, compression, and shear, and of the manner of transmission of stress. Figure 35 represents a pair of plates, each of breadth b and thickness t, transmitting a pull P in the direction of their length. The plates are united by means of a pin C, which fits tightly in a hole in the lower plate and passes

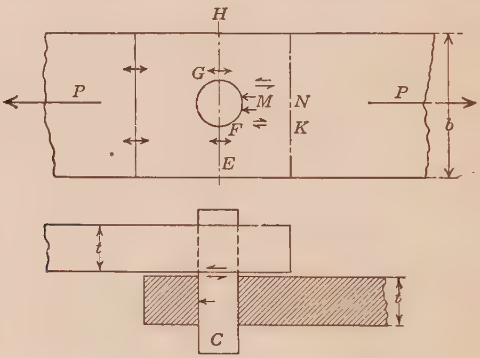


FIG. 35.—Stress at a bolted joint.

through a hole in the upper plate. The portion of the upper plate to the left of the pin is in tension. The intensity of this tensile stress is found by dividing the pull P by the area of the *gross section bt*. At the section EH in the plane of the axis of the hole, the stress is still tension. The area of this *net section* is smaller than that of the gross sections to the left and the unit stress is greater. If the hole is in the middle of the section and in the line of the pull, half of the total stress is transmitted by the section EF and half by the section GH. The stress which passes EF as tension passes FK as shear. The intensity of this shearing stress in the plate may be calculated by dividing the pull $\frac{P}{2}$ by the section of length FK and thickness t. At M, the surface of contact of the pin and plate, the stress is compression. The

force is transmitted as shearing stress from the portion of the pin in the upper plate to the portion in the lower plate, and, finally, as compression from the pin to the lower plate.

The stress may be regarded as flowing like an electric current. This is illustrated in Fig. 36. The circuit is completed through the bodies by which the force is applied.

If the pin in Fig. 35 is slightly smaller than the hole, all of the bearing pressure is applied to a narrow strip of the plate at M. The entire shearing stress is then transmitted by two sections extending from M to the edge of the plate at N. In calculating the shearing stress in the space between a rivet or bolt and the edge of the plate, it is customary to consider the minimum distance instead of the distance FK.

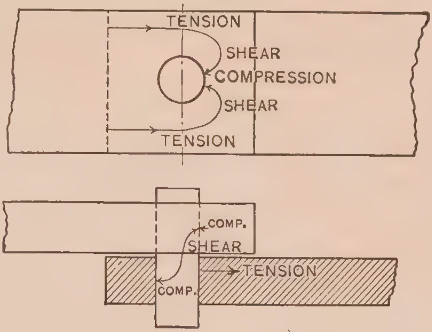


FIG. 36.—Flow of stress.

Problems

1. The plates in Fig. 36 are each 3 inches wide and ½ inch thick. The hole in the upper plate is 1 inch in diameter and that in the lower plate is ⅞ inch in diameter. The bolt is ⅞ inch in diameter and the total pull is 5,100 pounds. Find the unit shearing stress in the bolt and the unit tensile stress in the net section of each plate.

Ans. s_s, 8,480 pounds per square inch.
s_t, 5,100 pounds per square inch, net section, upper plate.
s_t, 4,800 pounds per square inch, net section, lower plate.

2. In Problem 1 find the unit shearing stress in the upper plate to right of the bolt if the center of the hole is 1.5 inches from the right edge of the plate. *Ans.* 5,100 pounds per square inch.

34. Bearing or Compressive Stress.—In calculating the unit bearing or compressive stress at the surface of contact of the pin and plate, it is customary, among engineers, to regard the bearing area as the product of the thickness of the plate multiplied by the diameter of the pin. If d is the diameter of the pin and t is the thickness of the plate, the *bearing area* is td. In other words, it is the projection upon a plane parallel to the axis of the pin of that portion of the pin which is inside of the plate.

Figure 37 shows a rectangular bar of thickness d, which is placed across the edge of a plate of thickness t. If the bar crosses

the plate at right angles, it is evident that the bearing area is td. If, as in Fig. 38, the bar passes through a hole in the plate, the bearing area is the same; and, if the forces P_1 and P_2 are balanced with respect to the center of the plate, the bearing stress is uniform over the entire area. If the forces are not balanced, the area and the *average* bearing stress remain the same,

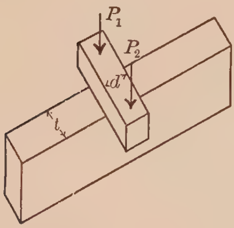


FIG. 37.—Bearing.

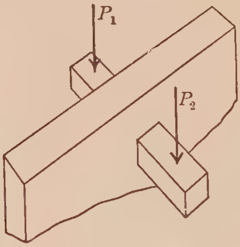


FIG. 38.—Bearing.

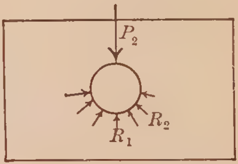


FIG. 39.—Bearing.

but the maximum bearing stress is greater. If there is force on the bar on one side of the plate only, the maximum bearing stress is much greater than the average.

Figure 39 shows a round bolt or pin passing through a plate. The actual area of contact is the lower half of the surface of the cylinder of length t and diameter d.

The reactions R_1, R_2, etc., are not all vertical, but are nearly normal to the surface of contact. If, as in the case of liquid pressure, these reactions were exactly normal and of equal intensity, the resultant of their vertical components would be the same as if that unit pressure were exerted on the horizontal projection of this cylindrical surface.

Problems

1. In Problem 1 of Article 33, what is the unit compressive stress of the pin?
Ans. 11,657 pounds per square inch.
2. Two ½-inch plates, each 4 inches wide, are united by a single 1-inch bolt. The unit tensile stress in the gross section of the plates is 4,000 pounds per square inch. Find the unit shearing stress in the bolt and the unit bearing stress between the bolt and the plates.
Ans. $s_s = 10{,}186$ pounds; $s_c = 16{,}000$ pounds.
3. Two steel plates, each approximately 4 inches wide and ¾ inch thick were united by three ⅞-inch rivets in a single row *lengthwise* the plates. (Watertown Arsenal, 1911, page 129.) When tested in tension, the first slipping occurred at a load of 45,200 pounds and the joint failed by shearing the rivets at 89,200 pounds. Find the unit tensile stress in the

net section of the plate, the unit bearing stress between the plates and the rivets, and the unit shearing *strength* of the rivets.

Ans. $s_t = 38{,}060$; $S_c = 45{,}300$; $s_s = 49{,}448$.

4. A joint similar to the one of Problem 3 had $1\frac{1}{8}$-inch rivets. First slipping occurred at a load of 48,900 pounds, and the joint failed by fracture of one plate at the outer rivet hole under a load of 128,500 pounds. Find the unit tensile *strength* at the net section, the unit bearing stress, and the unit shearing stress. *Ans.* $s_t = 59{,}590$ pounds per square inch.

35. Lap Joint with Single Row of Rivets.—Figure 40 shows a *lap joint* with a single row of rivets. In any riveted joint the distance p from center to center of adjacent rivets in a row is called the *pitch*. In solving problems, it is often convenient to consider a single strip of width equal to the pitch. The problem of a lap joint with a single row of rivets then becomes the same as that of Article 33. This strip may extend from center to center of adjoining rivets, as is shown between the two lower rivets of Fig. 40. The tension is transmitted by the net section between the rivets, and the shear is equally divided between the upper half of the lower rivet and the lower half of the second rivet. The unit strip may be taken to include one rivet, as is shown at the third rivet from the bottom of Fig. 40. The shear is then carried by the single rivet, while the tension is divided.

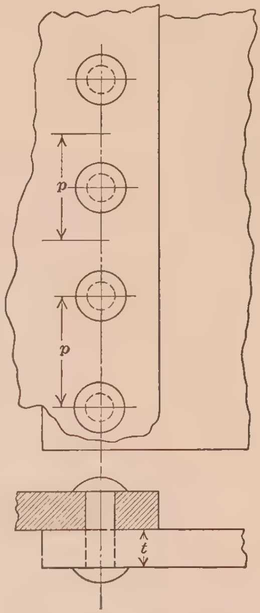


FIG. 40.—Single-riveted lap joint.

In the problems, unless otherwise stated, the rivet will be considered as entirely filling the rivet hole. In practice, when rivet holes are punched and not reamed, it is customary to make some allowance for the material near the hole which is weakened by overstrain. This allowance will not be made in the problems which follow. It is assumed that all rivet holes are reamed or drilled, and that the finished rivets exactly fit.

When the width of the plate is given, it is generally best to consider the entire plate as the unit.

Problems

(Look up rivet areas in handbook)

1. Two ½-inch plates, each 12 inches wide, are united by five ⅞-inch rivets in a single row to form a lap joint. The joint transmits a pull of 24,400 pounds. Find the unit tensile stress in the gross section and in the net section of the plates, the unit shearing stress in the rivets, and the unit compressive stress between the rivets and the plates.

$$\textit{Ans.} \begin{cases} s_t = 6{,}400 \text{ pounds per square inch in the net section;} \\ s_s = 8{,}116 \text{ pounds per square inch;} \\ s_c = 11{,}154 \text{ pounds per square inch.} \end{cases}$$

2. Two ⅜-inch boiler plates are united by a single row of $^{15}\!/_{16}$-inch rivets to form a lap joint. The pitch is 2⅜ inches. Find the unit tensile stress in the net section when the unit shearing stress in the rivets is 8,800 pounds per square inch. *Ans.* 11,269 pounds per square inch.

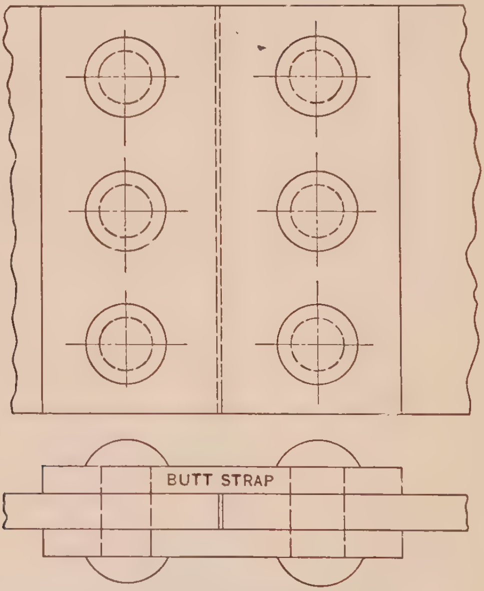


FIG. 41.—Single-riveted butt joint.

36. Butt Joint.—A butt joint is made when the two principal plates are in the same plane, and are united by means of one or two additional plates called *butt* straps. A butt joint with a single butt strap is equivalent to a pair of lap joints placed tandem.

Figure 41 shows a butt joint with double butt straps. Since the rivets are in double shear, the total shear transmitted by each rivet is twice as great as in a lap joint.

Example

Two ½-inch plates are united to form a butt joint by two 5⁄16-inch butt straps. There is one row of ⅞-inch rivets on each side. If the allowable unit tensile stress in the plate is 10,000 pounds per square inch, and the allowable unit shearing stress in the rivets is 8,000 pounds per square inch, what should be the pitch?

The area of one rivet is 0.6013 square inch, and each rivet is in double shear. The net cross-section which carries the tension equal to the shear in one rivet is $\frac{1}{2}\left(p - \frac{7}{8}\right)$ so that,

$$\frac{1}{2}\left(p - \frac{7}{8}\right) 10{,}000 = 2 \times 0.6013 \times 8{,}000,$$

$$p - \frac{7}{8} = 1.924 \text{ inches},$$

$$p = 2.80 \text{ inches}.$$

Problem

1. Two ⅜-inch plates are united by two ¼-inch butt straps to form a butt joint. There is one row of ¾-inch rivets on each side. The pitch is 2½ inches. Find the unit tensile stress in the net section of the ⅜-inch plates and the unit bearing stress between these plates and the rivets when the unit shearing stress in the rivets is 6,000 pounds per square inch.
Ans. s_t = 8,079 pounds per square inch.
s_c = 18,850 pounds per square inch.

Figure 42 represents a set of tests made at the Watertown Arsenal, to study the behavior of riveted joints. A plate of width b and thickness t was planed down for a portion of its length to some convenient width and united to a pair of plates, thus forming one-half of a butt joint. Wrought-iron rivets were used of nominal diameter 1⁄16 inch less than the diameter of the holes. In calculating it was assumed that the finished rivets entirely filled the rivet holes.

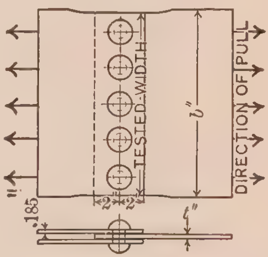


FIG. 42.—Half of butt joint.

Problems

2. In test piece No. 1,353 (Watertown Arsenal, 1885, page 867), the breadth b was 14.90 inches; the tested width, 14.39 inches; the actual thickness of the plate, 0.248 inch. There were five rivets in 1-inch drilled holes. The joint failed by tension along the line of the rivet holes under a pull of 156,440 pounds. The calculated results as published are:

Areas	Square inches
Gross sectional area of plate	3.569
Net sectional area of plate	2.329
Bearing surface of rivets	1.240
Shearing area of rivets	7.854

Maximum stress on joint	Pounds per square inch
Tension in gross section of plate	43,830
Tension in net section of plate	67,170
Compression on bearing surface of rivets	126,160
Shearing in rivets	19,920

Verify these results.

3. In test piece No. 1,355 the results were:

Tested width of plate	15 inches.
Actual thickness	0.251 inch.
Ultimate load	167,200 pounds.

There were five rivets in 1-inch holes. "Fractured two outside sections of plate at edge along line of riveting; the two middle sections sheared in front of the rivets."

Compute all unit stresses as in Problem 2.

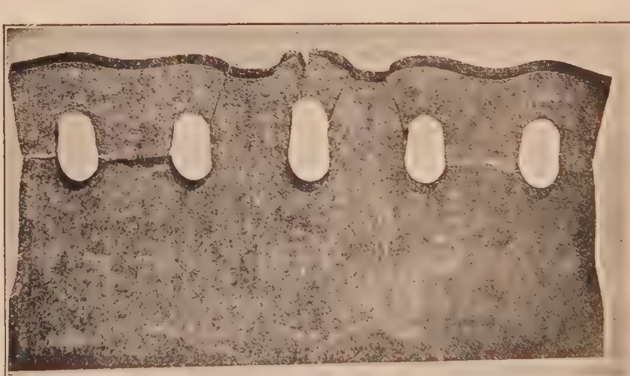


FIG. 43.—Failure of riveted plate.

Figure 43 is a copy of a photograph of this plate after failure. It shows failure by tension across the net section and shear in front of the rivets. It also shows elongation of the rivet holes due to bearing pressure on the plate, combined with shear.

In order to compare the strength of the material in the net section of a riveted joint with the ordinary tension tests, two strips were sheared from each sheet of steel, one lengthwise, the other crosswise the sheet. These were planed to a width of 1.5 inches and tested in the usual way.

From the sheet used in No. 1,353 two test pieces were taken. These gave as ultimate tensile strengths:

	Pounds per square inch
No. 1,213, lengthwise	59,180
1,224, crosswise	60,840

Four test strips were taken from the sheets used for No. 1,355:

	Pounds per square inch
No. 1,214, lengthwise	58,680
1,220, lengthwise	62,300
1,225, crosswise	61,230
1,226, crosswise	60,890

The ultimate strength of these test pieces was considerably smaller than the unit stresses in the net sections of the riveted plates which failed in tension. This difference is an illustration of Article 24. Since the net section of the riveted plate was relatively short, the material under the maximum stress was kept from necking by the material of the wider sections.

These tension tests show no definite difference between the strength of the strips taken lengthwise the plate and those taken crosswise the plate. This is explained by the fact that the plates were rolled in both directions. When rolled metal is worked in one direction only, its tensile strength is greater in that direction.

Problems

4. In a test piece similar to Fig. 43 (Watertown Arsenal, 1886, page 1401), the following data are given; tested width, 13.11 inches; thickness, 0.630 inch; five rivets in 1-inch drilled holes; failed by shearing the rivets under a pull of 295,500 pounds; rivet holes elongated 0.31 inch, 0.32 inch, 0.26 inch, 0.25 inch, 0.24 inch.

Calculate the unit stresses.

		Pounds per square inch
Ans.	Tensile stress in net section	57,840
	Bearing stress	93,810
	Shearing stress on the rivets	37,620

5. In Problem 4, the butt straps were 0.384 inch thick. Find the unit tensile stress in the net section.

6. Two plates, *A* and *B*, each approximately 4 inches wide and 1.5 inches thick, were united to form a butt joint with two ¾-inch butt straps. There were three 1⅛ rivets on each side arranged in a single row length-

wise the plate. When the joint was tested at Watertown Arsenal (1911, page 129) the first slipping occurred at a load of 124,500 pounds. The plate B scaled at 155,300 pounds. The plate A fractured across the first rivet hole at 223,200 pounds. Find the shearing stress, the bearing stress, the ultimate tensile strength at the net section, and the yield point.

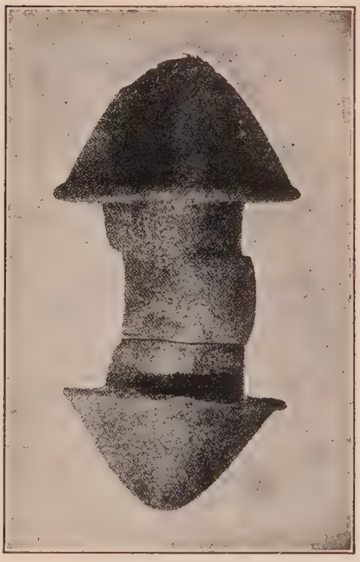


FIG. 44.—Failure of rivet.

Figure 44 is a copy of a photograph of a rivet which failed by shear as in Problem 4 (Watertown Arsenal, "Tests of Metals," 1886, page 1567).

37. Rivets in More Than One Row.—Rivets are frequently arranged in two or more rows. The rivets in the second row may be placed directly behind the rivets in the first row, or they may be arranged zigzag as shown in Fig. 45. Two adjoining rows of zigzag rivets must not be placed too close together or the plate will fail along the diagonal line joining the rivets of the two rows. The Boiler Code of the American Society of Mechanical Engineers specifies that the minimum distance between rows of rivets (called *back pitch*) shall be twice the diameter of the rivet holes, when the pitch is not more than four times the diameter of the rivet holes, and the rivets are arranged as in Fig. 45.*

In computing problems of two or more rows of rivets it is customary to regard the unit shearing stress the same in all rivets.

Where narrow plates are united by several rows of rivets, it is best to take the entire width of the plate as the unit.

Example

Two ¾-inch by 12-inch plates are united to form a lap joint by means of ten 1-inch rivets arranged as shown in Fig. 46. The joint transmits a pull of 60,000 pounds. Find the unit shearing stress in the rivets and the unit tensile stress in plate A at sections 1–1, 2–2 and 3–3.

$$s_s = \frac{60{,}000}{10 \times 0.7854} = 7{,}639 \text{ pounds per square inch.}$$

* See Report of Boiler Code Committee, American Society of Mechanical Engineers, page 44, Edition 1918, for full specifications.

At section 1–1 the net area is (12 − 1)¾ and

$$s_t = \frac{60{,}000}{11 \times \frac{3}{4}} = 7{,}273 \text{ pounds per square inch.}$$

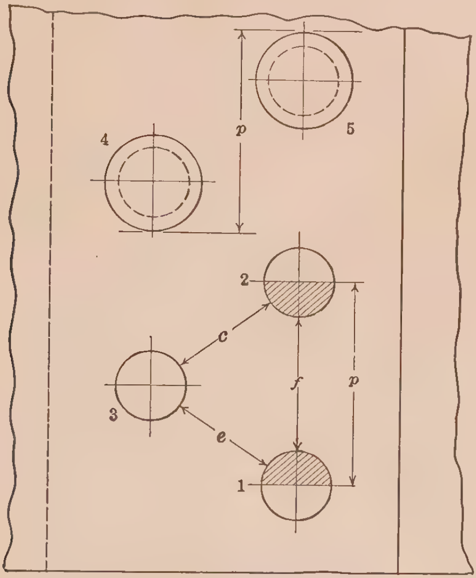


FIG. 45.—Double-riveted lap joint.

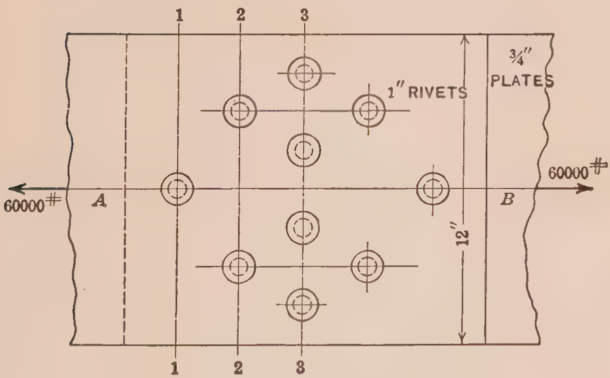


FIG. 46.—Multiple-riveted lap joint.

At section 2–2 the net width is 10 inches, but since one-tenth of the total pull has been transmitted by rivet 1 from plate A to plate B the total tension transmitted through this net section is only 54,000 pounds, and

$$s_t = \frac{54{,}000}{10 \times \frac{3}{4}} = 7{,}200 \text{ pounds per square inch.}$$

At section 3–3 the net width is 8 inches, but three-tenths of the total pull has been transmitted to plate B through the rivets in sections 1–1 and 2–2 so that the total tension in net section 3–3 is only 42,000 pounds.

$$s_t = \frac{42{,}000}{6} = 7{,}000 \text{ pounds per square inch.}$$

Problem

1. Solve the above example if the plates are 10 inches wide instead of 12 inches.

$$\textit{Ans.} \begin{cases} \text{At 1–1, } s_t = 8{,}889 \text{ lb./in.}^2 \\ \text{At 2–2, } s_t = 9{,}000 \text{ lb./in.}^2 \\ \text{At 3–3, } s_t = 9{,}333 \text{ lb./in.}^2 \end{cases}$$

Notice that in Problem 1 the greatest tensile stress is at section 3–3, while in the example the greatest stress is at section 1–1.

In wide plates, such as are used in boilers, it is not convenient to consider the entire width, but it is better to divide the width up into a number of equal units, each of which includes a group of rivets. Generally the width of such a unit is the pitch in the row having the fewest rivets.

In Fig. 45, where the pitch of both rows is the same, the width of the unit is equal to the pitch. It may extend from center to center of two rivets in one row. The entire pull which is transmitted by a strip of the plate of width p may be regarded as carried by the whole of rivet 3, the shaded portion of rivet 2, and the shaded portion of rivet 1; or a strip of equal width may include all of rivets 4 and 5. In either case, one rivet in each row is included in the unit. All the stress which is transmitted by a strip of unit width passes through the net section of the lower plate between two rivets of the right row. Half of this stress is transmitted to the upper plate through the rivet of the right row, and the other half passes through the net section at the left row of rivets. The unit tensile stress in the net section at the right row in the lower plate and at the left row in the upper plate is twice as great as the unit stress in the left row in the lower plate and the right row in the upper plate.

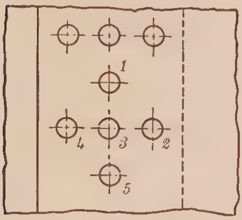


Fig. 47.—Rivets in three rows.

Figure 47 shows three rows of rivets with twice as many rivets in the middle row as in either of the others. The unit strip is taken as equal to the pitch in the outer rows. The unit strip may extend from the center of rivet 1 to the center of rivet 5, or it may include the whole of rivet 1 and none of rivet 5. In

either case it embraces two rivets in the middle row and one rivet in each of the others.

Problems

2. Two $\frac{5}{16}$-inch plates are united by $\frac{3}{4}$-inch rivets to form a lap joint. The rivets are in three rows as in Fig. 47 with the long pitch 5 inches. The gross section of the plates transmits a pull of 2,720 pounds per inch of width. Find all the stresses.

		Pounds per square inch
Ans.	s_s in all rivets	7,696
	s_c	14,507
	s_t right upper and left lower	10,240
	s_t left upper and right lower	2,560
	s_t at middle rows	9,326

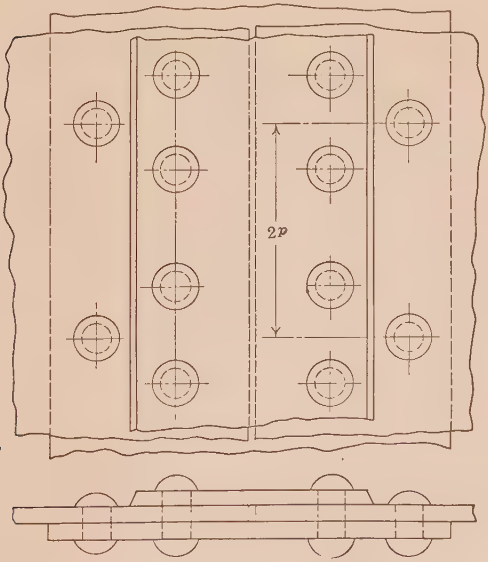


FIG. 48.—Double-riveted butt joint.

3. A butt joint is formed of two $\frac{1}{2}$-inch plates united by two $\frac{3}{8}$-inch butt straps. There are two rows of 1-inch rivets on each side. The pitch of the inner row is 3 inches and that of the outer row is 6 inches. The unit stress in the gross section of the plate is 8,000 pounds per square inch. Find the unit tensile stress at the net section at each row of rivets in the $\frac{1}{2}$-inch plates and at the inner rows in the butt straps. Find the unit shearing stress in the rivets and the unit bearing stress between the rivets and the $\frac{1}{2}$-inch plates.

		Pounds per square inch
Ans.	s_t in ½-inch plates at outer rows	9,600
	s_t in ½-inch plates at inner rows	8,000
	s_t in butt straps at inner rows	8,000
	s_s in all rivets	5,093
	s_c between rivets and ½-inch plates	16,000

4. Solve Problem 3 if one of the butt straps includes only the two inner rows of rivets while the other butt strap takes in all the rivets, Fig. 48.

		Pounds per square inch
Ans.	s_t in ½-inch plates at outer rows	9,600
	s_t in ½-inch plates at inner rows	9,600
	s_t in wide butt strap at inner row	9,600
	s_t in narrow butt strap at inner row	6,400
	s_s in all rivet sections	6,112
	s_c in inner rows	19,200
	s_c in outer rows	9,600

38. Efficiency of a Riveted Joint.—The ratio of the strength of a riveted joint to the strength of one of the plates which it unites is called the *efficiency* of the joint. The efficiency may also be defined as the ratio of the unit stress in the gross section, when the joint is stressed to its allowed limit, to the allowable unit stress in the plates. If the joint is so designed as to make it at least as strong in shear and compression as it is in tension at the net section, the efficiency becomes the ratio of the net section to the gross section.

The calculations for efficiency may be based upon either the ultimate strength or the allowable unit stress. The Boiler Code of The American Society of Mechanical Engineers specifies:

	Pounds per square inch
Tensile strength of steel plate	55,000
Crushing strength of steel plate	95,000
Shearing strength of steel rivets	44,000
Shearing strength of iron rivets	38,000

The code further specifies that the strength of a rivet in double shear is twice its strength in single shear.

With a factor of safety of 5, which is the one generally used in boiler design, the allowable unit stresses become:

	Pounds per square inch
s_t for steel	11,000
s_c for steel	19,000
s_s for steel	8,800
s_s for iron	7,600

To find the efficiency of a riveted joint, calculate the strength of unit width in tension at the net sections, in shear at the rivets, in bearing between the rivets and the plates, and in all possible combinations of these and divide the smallest of these values by the tensile strength of the unit width at the gross section.

The design of joints of maximum efficiency is discussed in Appendix *A*.

Example I

Two ⅜-inch steel plates are united by a double row of ¾-inch steel rivets to form a lap joint. The pitch in each row is 2¾ inches. Using the ultimate strengths of the A. S. M. E. Boiler Code,* calculate the efficiency of the joint.

The unit strip is 2¾ inches wide, and the net width between rivets is 2 inches.

	Pounds
Tensile strength of net section $= 2 \times \frac{3}{8} \times 55{,}000$...	41,250
Shearing strength of rivets $= 2 \times 0.4418 \times 44{,}000$...	38,878
Bearing strength $= 2 \times \frac{3}{8} \times \frac{3}{4} \times 95{,}000$..........	53,437
Tensile strength of unit $= 2\frac{3}{4} \times \frac{3}{8} \times 55{,}000$.......	56,719

$$\text{Efficiency} = \frac{38{,}878}{56{,}719} = 0.685 = 68.5 \text{ per cent.}$$

Problems

1. Two ⅜-inch plates are united by a double row of 1-inch rivets to form a lap joint. The pitch is 4½ inches. Find the strength of unit strip of the joint and the efficiency.

		Strength of unit strip in pounds
Ans.	Tension in gross section....................	92,812
	Tension in net section......................	72,187
	Shear in rivets............................	69,115
	Bearing..................................	71,250
	Efficiency................................	74.5 per cent.

2. Two ½-inch plates are united to form a butt joint, with two rows of ¾-inch rivets on each side. The pitch of outer rows is 4½ inches and that of the inner rows is 2¼ inches. Find the efficiency.

Ans. Weakest in tension. Efficiency, 83.3 per cent.

3. Solve Problem 2 if the pitch of the outer rows is 5 inches and that of the inner rows is 2½ inches.

Ans. Weakest in compression. Efficiency, 77.7 per cent.

4. Solve Problem 2 if the thickness of the plates is ⅝ inch instead of ½ inch.

Ans. Weakest in shear. Efficiency, 75.4 per cent.

* *Transactions of the American Society of Mechanical Engineers*, 1914.

Example II

In Fig. 48, the pitch of the outer rows is 5 inches, the pitch of the inner rows is $2\frac{1}{2}$ inches, the diameter of the rivets is $\frac{3}{4}$ inch, the thickness of the plates is $\frac{3}{8}$ inch, and the thickness of the butt straps is $\frac{5}{16}$ inch. Find the strength and efficiency of the joint.

In a unit strip 5 inches wide there are two rivets in double shear and one rivet in single shear.

	Pounds
Shearing strength of one rivet section $= 0.4418 \times 44{,}000$	19,439
Bearing of one rivet on plate $= \frac{3}{8} \times \frac{3}{4} \times 95{,}000$	26,718
Bearing of one rivet on butt strap $= \frac{5}{16} \times \frac{3}{4} \times 95{,}000$	22,265
Total shearing strength $= 5 \times 19{,}439$	97,195
Tension in outer net section $= \frac{3}{8} \times \frac{17}{4} \times 55{,}000$	87,656
Tension in inner net section $= \frac{3}{8} \times \frac{7}{2} \times 55{,}000$	72,187
Tension in gross section $= \frac{3}{8} \times 5 \times 55{,}000$	103,125

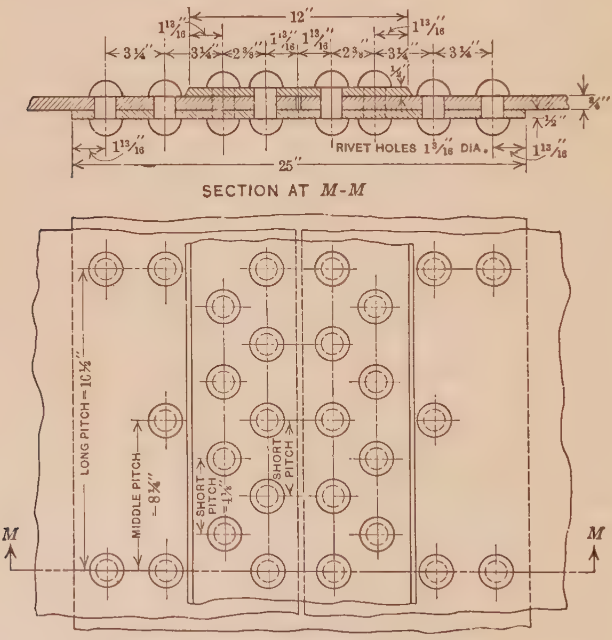


FIG. 49.—Quadruple-riveted butt joint.

The joint may fail by tension at inner section with shear or compression at outer rivet.

Tension at inner section and shear $= 72{,}187 + 19{,}439$	91,626

The joint may fail by compression at the two inner rivets and shear at the outer rivet.

$26{,}718 \times 2 + 19{,}439$	72,875

Since the compressive strength of the outer rivet is greater than the shearing strength, it is not necessary to calculate the total strength in compression or the effect of tension at the inner section with compression at the outer rivet. Since the bearing strength of the butt strap against outer rivet is greater than the shearing strength of one rivet in single shear, it is not necessary to consider any combination of this bearing strength with the strength at the inner row.

$$\text{Efficiency} = \frac{72{,}875}{103{,}125} = 70.7 \text{ per cent.}$$

Problems

5. Solve Example II with ¼-inch butt straps.
6. Solve Example II with 5⁄16-butt straps but with ⅞-inch rivets.
7. Figure 49 shows one of a series of quadruple-riveted butt joints designed by the Hartford Boiler Inspection and Insurance Co. The short pitch is 4⅛ inches and the rivet holes are 13⁄16 inch. Find the efficiency, using the unit stresses of the A. S. M. E. Boiler Code.

Ans. 92.7 per cent. Joint fails by tearing the plate between rivet holes in the third row and shearing the rivets in the two outer rows.

39. Circumferential Stress in Hollow Cylinders.—In a hollow vessel inclosing a liquid or gas under pressure, the pressure of the fluid develops stresses in the walls of the vessel. The pressure of a fluid at any point is normal to the surface. The result-

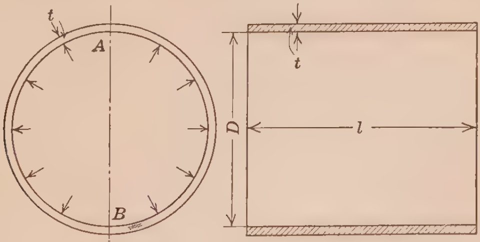


FIG. 50.—Hollow cylinder with internal pressure.

ant pressure on any portion of a curved surface in any *given direction* is equal to the pressure on a plane surface perpendicular to the given direction and equal in area to the projection of the curved surface upon its plane. Figure 50 represents a portion of the surface of a cylinder of diameter D, and of length l perpendicular to the plane of the paper. If P is the pressure on this surface in pounds per square inch, the total pressure on the semicircular surface to the right of the plane AB is $\frac{P\pi Dl}{2}$. The resultant pressure on this surface in the direction normal to AB

is PDl, since Dl is the area of the projection of the curved surface upon the vertical plane. There is an equal pressure in the opposite direction upon the curved surface to the left of AB. These two equal and opposite forces are resisted by the circumferential tensile stresses in the sections at A and B. If t is the thickness of the wall of the cylinder, the area in tension is $2tl$, and

$$2tls_t = PDl, \tag{1}$$

$$s_t = \frac{PD}{2t}. \qquad \text{Formula V.}$$

Problems

1. A boiler shell 30 inches in diameter is subjected to a steam pressure of 150 pounds per square inch. The plates are ⅜ inch thick. Find the unit stress. *Ans.* 6,000 pounds per square inch.
2. A boiler shell 5 feet in diameter is made of 9/16-inch plates. The longitudinal joints have an efficiency of 80 per cent. The allowable unit tensile stress in the plates is 11,000 pounds per square inch. Find the allowable pressure. *Ans.* 165 pounds per square inch.

40. Longitudinal Stress in a Hollow Cylinder.—The resultant pressure on the end of a hollow cylinder in the direction of its length is the product of the pressure in pounds per square inch multiplied by the area of cross-section of the cylinder. This is resisted by a longitudinal tensile stress in the shell. The cross-section in tension is approximately the inner circumference multiplied by the thickness.

$$s_t\pi Dt = \frac{P\pi D^2}{4}, \tag{1}$$

$$s_t = \frac{PD}{4t}. \tag{2}$$

The longitudinal unit stress in a hollow cylinder is one-half the circumferential unit stress.

Equations (1) and (2) apply also to hollow spheres subjected to internal pressure.

Since the longitudinal tensile stress is only one-half the circumferential unit stress, it is only necessary to have the efficiency of the circumferential joints which resist the longitudinal tension a little greater than one-half the efficiency of the longitudinal joints.

Problems

1. A hollow cylinder, 4 feet in diameter, is made of $\frac{3}{8}$-inch plates. The pressure is 120 pounds per square inch. Find the longitudinal stress.

Ans. 3,840 pounds per square inch.

2. A sphere is 10 inches inside diameter and 10.2 inches outside diameter. The allowable unit stress is 10,000 pounds per square inch. What is the allowable internal pressure?

3. A boiler 40 inches in diameter is made of $\frac{5}{16}$-inch plates. The longitudinal joints are triple-riveted butt joints with an efficiency of 82 per cent. The circumferential joints are single riveted lap joints of 1-inch rivets spaced 3 inches apart. With the A. S. M. E. ultimate strengths and a factor of safety of 5, what is the allowable pressure?

CHAPTER V

TORSION

41. Torque.—A shaft or rod subjected to a pair of equal opposite couples which are in parallel planes at right angles to its length is in *torsion* between these planes. Figure 51 shows a horizontal shaft which is supported by two bearings and carries two pulleys. A rope is wound a part of the way around each pulley and fastened to it. Each rope supports a load. The load P on the smaller pulley and part of the reactions of the bearings form a counterclockwise couple. The load W on the larger pulley and a part of the reactions form a clockwise couple. If there is no friction at the bearings, these opposite couples are equal, provided the shaft is stationary or moving in either direction with constant speed. The moment of either couple is the twisting moment or torque in the portion of the shaft between the two pulleys. Torque, which is represented in algebraic equations by T, is expressed in foot-pounds or inch-pounds. In order to distinguish torque and bending moment from work, some writers use *pound-feet* and *pound-inches* for the first two, and reserve foot-pounds and inch-pounds to mean work or energy. This distinction, however, is not generally made.

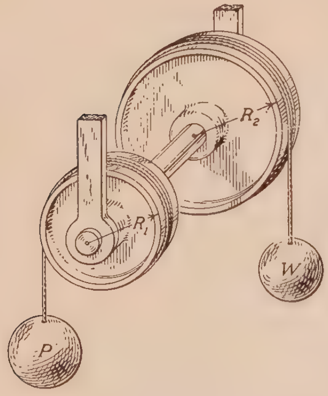


Fig. 51.

Problems

1. In Fig. 51, the diameter of the smaller pulley is 36 inches. The load P of 400 pounds is hung on a 1-inch rope. Find the torque.

   Ans. 7,400 inch-pounds.

2. The load W of Fig. 51 hangs on a 1-inch rope and exactly balances the load of Problem 1. If the diameter of the larger pulley is 4 feet what is the weight of W?

3. A horizontal shaft carries a crank which is 20 inches long from axis of the shaft to the center of the crank pin. A vertical force of 600 pounds is applied to the crank pin. Find the torque when the crank makes an angle of 30 degrees with the horizontal.
4. A shaft carries a 6-foot pulley. A belt over this pulley exerts a pull of 2,000 pounds at one point of tangency and 300 pounds at the other. Find the torque if the thickness of the belt is neglected.

42. Deformation and Stress at Surface of Shaft.—Figure 52 represents a shaft fixed at the lower end. DB and EF are lines in its surface parallel to the axis CO. If the cylindrical surface

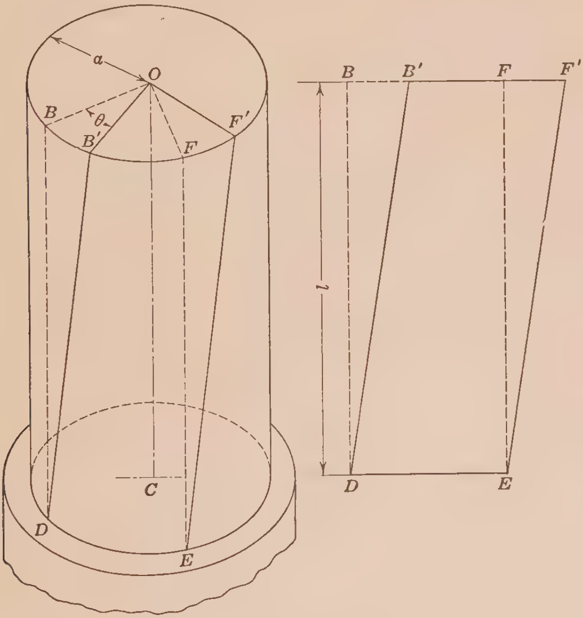


FIG. 52.—Portion of shaft in torsion.

between the lines DB and EF is developed, it forms the plane rectangle $DBFE$. If a torque is applied to the shaft, twisting it in a counterclockwise direction, the point B is displaced to B' and the point F is displaced to F'. The developed surface $DBFE$ suffers a shearing deformation and becomes the parallelogram $DB'F'E$. Every point on the surface at the upper end is displaced the distance BB'. If a is the radius of the cylinder and θ (in radians) is the angle through which the top is turned with reference to the base, the displacement BB' is equal to $a\theta$.

If l is the length of the shaft, the unit shearing deformation is given by

$$\delta_s = \frac{a\theta}{l}, \tag{1}$$

and the unit shearing stress in the outer fibers is given by

$$S_s = \frac{E_s a\theta}{l}. \tag{2}$$

Problems

1. A 6-inch solid shaft is twisted 3 degrees in a length of 15 feet. What is the unit shearing displacement at the surface?
Ans. $\delta_s = \frac{\pi}{3{,}600} = 0.000873.$
2. If the modulus of rigidity of the material of Problem 1 is 11,500,000 pounds per square inch, what is the unit shearing stress in the outer fibers?
Ans. $S_s = 10{,}040$ pounds per square inch.
3. What is the angle of twist in a 10-foot length of 1-inch rod when the unit shearing stress in the outer fibers is 8,000 pounds per square inch, if modulus of rigidity is 11,000,000 pounds per square inch?
Ans. $\theta = 10$ degrees nearly.
4. A ½-inch rod is twisted 36 degrees in a length of 15 feet. What is the unit displacement in the outer fibers?
5. How much is a 2-inch rod twisted in a length of 5 feet if the unit displacement is the same as in Problem 4?

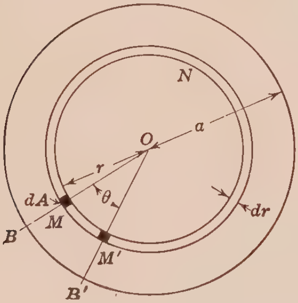


FIG. 53.—Shear displacement of torsion.

43. Relation of Torque to Angle of Twist.—Figure 53 shows the upper end of the shaft of Fig. 52. An element of area dA is at the position M at a distance r from the axis. When the shaft is twisted and the top is turned through an angle of θ *radians*, this area is moved to M'. Its displacement is $r\theta$ and the unit shearing displacement is given by

$$\delta_s = \frac{r\theta}{l}. \tag{1}$$

The unit shearing stress on dA is given by

$$s_s = \frac{E_s r\theta}{l}. \tag{2}$$

The part of the shaft between the radius r and the radius $r + dr$ is a hollow cylinder of thickness dr, which may be developed into a rectangular solid of width $2\pi r$ and thickness dr. The area of cross-section of this hollow cylinder is $2\pi r dr$. The shearing force required to deform this cylinder is the product of its cross-section multiplied by the unit shearing stress.

$$\text{Shearing force} = 2\pi r dr \times \frac{E_s r\theta}{l} = \frac{2\pi E_s \theta}{l} r^2 dr. \tag{3}$$

The moment of this shearing force with respect to the axis of the cylinder is the product of the force by the distance r.

$$\text{Resisting moment} = \frac{2\pi E_s \theta}{l} r^3 dr. \tag{4}$$

The entire shaft may be regarded as made up of a series of concentric hollow cylinders of thickness dr, and the total resisting moment, which is equivalent to the external torque, is the integral of Equation (4) between the limits $r = 0$ and $r = a$.

$$T = \frac{2\pi E_s \theta}{l} \int r^3 dr = \frac{\pi E_s \theta}{2l} [r^4]_0^a;$$

$$T = \frac{E_s \theta}{l} \frac{\pi a^4}{2}. \tag{5}$$

The expression $\frac{\pi a^4}{2}$ is the polar moment of inertia of the circle of radius a and is usually represented by J. Equation (5) becomes

$$T = \frac{E_s \theta J}{l}, \tag{6}$$

from which $$\theta = \frac{Tl}{E_s J}.$$ Formula VI.

This theory applies rigidly to circular shafts only. In Fig. 53, the straight line OMB remains straight when the shaft is twisted, provided the sections are circular. When the sections are not circular, a straight line from the center to the surface does not remain straight when torque is applied. The unit stress is not, therefore, proportional to the distance from the axis, and the equations above are not valid.

Problems

1. A 2-inch solid steel shaft is twisted 3 degrees in a length of 12 feet. If the modulus of rigidity is 12,000,000 pounds per square inch, what is the torque?

   Ans. 6,854 inch-pounds.

2. Find the angle of twist in a 15-foot length of a 6-inch solid shaft, with a 6-foot pulley, if the tension on the belt at the one point of tangency is 3,500 pounds and at the other point of tangency is 300 pounds. The modulus of rigidity is 11,400,000 pounds per square inch.

   Ans. 0.819 degree.

3. A hollow tube, 3 inches outside diameter and 2 inches inside diameter, is twisted 1 degree in a length of 40 inches by a force of 520 pounds at the end of an arm 5 feet in length. Find E_s.

   Ans. $E_s = 11{,}205{,}000$ pounds per square inch.

4. A hollow tube of the same material as that of Problem 3 is 5 inches outside diameter and 3 inches inside diameter. How much would it be twisted in a length of 20 feet by a force of 2,080 pounds at the end of an arm 8 feet long?

44. The Relation of Torque to Shearing Stress.—From Equation (6) of Article 43,

$$T = \frac{E_s\theta J}{l}; \tag{1}$$

and from Equation (2) of Article 43,

$$S_s = \frac{E_s a\theta}{l}; \tag{2}$$

from which

$$\frac{E_s\theta}{l} = \frac{S_s}{a}. \tag{3}$$

Substituting in (1)

$$T = \frac{S_s J}{a}, \tag{4}$$

$$S_s = \frac{Ta}{J} \qquad \text{Formula VII.}$$

Formula VII may be derived by another method, which is not based on the formulas of the preceding articles. Figure 53 shows that the element dA is displaced a distance $r\theta$ when the shaft is twisted. The displacement is proportional to r. The unit displacement, and, consequently, the unit stress is proportional to r, which is the distance of the element dA from the axis of the shaft.

If s_1 is the unit shearing stress at unit distance from the axis, the unit shearing stress at a distance r is $s_1 r$. The shearing stress

on an area dA is $s_1 r dA$ and the resisting moment is $s_1 r^2 dA$. The total moment is given by

$$T = s_1 \int r^2 dA; \tag{5}$$

Since $\int r^2 dA$ is the polar moment of inertia, which is represented by J,

$$T = s_1 J. \tag{6}$$

Since s_1 is the unit stress at unit distance from the axis, the unit stress at a distance r from the axis is $s_1 r$, and the unit stress at the surface at a distance a from the axis is given by

$$S_s = s_1 a; \tag{7}$$

from which

$$s_1 = \frac{S_s}{a}. \tag{8}$$

Substituting in Equation (6),

$$T = s_1 J = \frac{S_s J}{a}. \qquad \text{Formula VII.}$$

Problems

1. A 2-inch solid shaft is twisted by a force of 240 pounds applied at the end of an arm 4 feet in length. Find the unit shearing stress at the surface of the shaft.

$$\textit{Ans.}\ S_s = \frac{11{,}520 \times 2}{\pi} = 7{,}334 \text{ pounds per square inch.}$$

2. A 6-inch solid shaft exerts a torque of 30,000 foot-pounds. Find the maximum unit shearing stress.

   Ans. $S_s = 8{,}488$ pounds per square inch.

3. Solve Problem 2 if the shaft is hollow with inside diameter 2 inches. Find the unit shearing stress at the outer and at the inner surface.

   S_s at outer surface $\doteq 8{,}594$ pounds per square inch;
   s_s at inner surface $= 2{,}865$ pounds per square inch.

4. Solve Problem 2 if the shaft is hollow with inside diameter 2 inches and outside diameter such that the area of cross-section is the same as that of a 6-inch solid shaft.

5. What is the ratio of the weight of a hollow shaft 2 inches inside diameter and 6 inches outside diameter to the weight of a solid shaft 6 inches in diameter? What is the ratio of the polar moments of inertia?

6. If a is the outside radius of a shaft and b is the inside radius, what is the expression for the shearing stress at the surface?

$$\textit{Ans.}\ S_s = \frac{2Ta}{\pi(a^4 - b^4)}.$$

7. Derive an expression for the shearing stress at the surface of a solid shaft in terms of the diameter.

$$Ans.\ S_s = \frac{16T}{\pi d^3}.$$

45. Relation of Torque to Work.—To an arm of length R, measured from the axis of a shaft, a force P is applied which is perpendicular to the plane passing through the axis of the shaft and the point of application of the force. The torque is RP. When the shaft makes one revolution, the point of application of the force moves through a distance $2\pi R$. The work done by the force P is $2\pi RP$. Since PR is the torque,

$$\text{work} = 2\pi RP = 2\pi T.$$

The work per revolution is 2π times the torque. This relation is true, whether the torque is due to a single force or to a number of forces.

In problems relating to the work done by a rotating body, solve first for the torque. When this is obtained, it may be used in Formulas VI and VII.

Problems

1. A shaft transmits 720 horsepower at 200 revolutions per minute. Find the torque in foot pounds. *Ans.* 18,907 foot-pounds.
2. What must be the diameter of the shaft of Problem 1, if the shearing stress is 6,000 pounds per square inch? *Ans.* 5.77 inches.
3. How many horsepower may be transmitted by a hollow shaft, which is 6 inches inside diameter and 12 inches outside diameter, when the speed is 120 revolutions per minute and the shearing stress is 6,000 pounds per square inch? *Ans.* 3,634 hp.
4. If S_s is the allowable unit shearing stress, N is the number of revolutions per minute, hp is the horsepower, and a is the radius of the shaft, show that

$$a^3 = \frac{33{,}000 \times 12 \times hp.}{\pi^2 NS_s} = \frac{40{,}123\ hp.}{NS_s}.$$

5. If the allowable unit shearing stress is 5,000 pounds per square inch, show that the diameter of a solid shaft should be approximately,

$$d = 4\sqrt[3]{\frac{hp}{N}}.$$

46. Helical Springs.—An interesting illustration of torsion is the elongation or compression of a helical spring, such as is shown in Fig. 54. A helical spring is made by winding a wire or rod on a cylinder (in a single layer, usually). The radius of

the coil of the spring is the sum of the radii of the wire and the cylinder about which it is wound. When the spring is to be used in tension, the ends of the wire are turned in to the center, in order that the force may be applied in the line of the axis. Figure 54, II, is a plan of the lower turn. The force P is normal to the plane of the paper. Any portion of the spring CBO may be considered as a free body. The section at O is perpendicular to the wire. The plane of this section passes through the center C. The force P at C has no bending moment with respect to the section at O. The effect of the force P acting on the arm CBO is independent of the form of the arm. As far as the stresses at O are concerned, CBO might be a straight rod from C to O. The effect of the force P on the section at O is a torque PR. Since O is any point on the helix, the entire wire, except the portion CB and a similar portion at the top, is subjected to the torque PR. In addition to this torsion, there is a constant total shear P. Since the coils are not exactly horizontal, there is another slight correction. Both of these, however, are neglected in ordinary calculations.

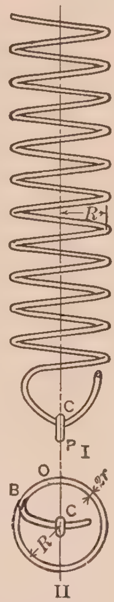


FIG. 54.—Helical spring.

The total elongation of a helical spring is calculated by multiplying the angle of twist in the entire length of the wire by the radius of the coil.

Problems

1. A rod 0.2 inch in diameter is used to make a helical spring of 20 turns. The radius of the coil from the axis to the center of all sections is 1 inch. What is the elongation, due to a load of 3 pounds, if the modulus of rigidity is 12,000,000 pounds per square inch.

$$T = 3 \text{ inch-pounds}; J = \frac{0.0001\pi}{2}; \text{ length of rod, } 40\pi.$$

$$\theta = \frac{3 \times 40\pi \times 2}{12{,}000{,}000 \times 0.0001\pi} = 0.2 \text{ radian.}$$

$$\text{Elongation} = 0.2 \times 1 = 0.2 \text{ inch.}$$

2. What is the unit shearing stress in Problem 1?

Ans. $S_s = \dfrac{6{,}000}{\pi} = 1{,}910$ pounds per square inch.

3. If the same rod were used to make a spring of ten turns, each of 2-inch radius, what would be the elongation due to a load of 3 pounds, and what would be the unit shearing stress?

Ans. 0.8 inch, 3,819 pounds per square inch.

4. At Watertown Arsenal, a steel rod 1.24 inches in diameter and about 241 inches long was formed into a helical spring 7.64 inches *outside* diameter. A load of 5,000 pounds shortened this spring 4.64 inches. Find the modulus of shearing elasticity *Ans.* 11,460,000.

5. In Problem 4 find the unit shearing stress under the load of 5,000 pounds.

Ans. 42,740 pounds per square inch.

6. If R is the radius of the helix, r the radius of the rod, P the load, E_s the modulus of elasticity in shear, and n the number of turns, prove that

$$\text{Elongation} = \frac{4PR^3n}{E_s r^4}.$$

7. If S_s is the allowable unit shearing stress, find the elongation of a spring in terms of S_s, E_s, R, r, and n.

Ans. $\text{Elongation} = \dfrac{2\pi S_s R^2 n}{E_s r}$.

8. Find the expression for the elongation of a helical spring in terms of S_s, E_s, R, r, and l, in which l is the length of the rod.

Ans. $\text{Elongation} = \dfrac{S_s R l}{E_s r}$.

47. Resilience in Torsion.—A force P at the end of an arm R twists a shaft of length l through an angle of θ radians. If there is no torque at the beginning, the average force is $\dfrac{P}{2}$, and the work of twisting is given by

$$\text{Work} = \frac{PR\theta}{2} = \frac{T\theta}{2} = U, \tag{1}$$

$$U = \frac{T^2 l}{2E_s J}. \tag{2}$$

If the shaft is circular and of radius a,

$$U = \frac{J^2 S_s^2 l}{2a^2 E_s J} = \frac{J S_s^2 l}{2a^2 E_s}. \tag{3}$$

For a solid circular shaft

$$U = \frac{\pi a^2 l S_s^2}{4E_s} = \frac{S_s^2}{4E_s} \times \text{volume}; \tag{4}$$

and the energy per unit volume $= U = \dfrac{S_s^2}{4E_s}$. Formula VIII.

Since the modulus of rigidity of metals is much less than the modulus of elasticity in tension or compression, the energy of torsion is relatively large. When Poisson's ratio is one-fourth, E_s is $\frac{2E}{5}$. When this value of E_s is substituted in Formula VIII the result is

$$U = \frac{5S_s^2}{8E}. \tag{5}$$

The elastic limit in shear is somewhat smaller than the elastic limit in tension or compression. If the limits were the same, a solid cylinder in torsion would store more energy than a block under direct stress. With direct tension or compression, the force must be very large and the displacement must be very small. With torsion, as in a helical spring, the displacement may be large and the force correspondingly small.

Problems

1. In Problem 4 of Article 46, find the work done by the load of 5,000 pounds in shortening the spring and the work per cubic inch.

Ans. 11,600 inch-pounds;
39.8 inch-pounds per cubic inch.

2. Find the resilience per cubic inch in Problem 1 by means of Formula VIII and the answers of Problems 5 and 4 of Article 46.

3. A spring at the Watertown Arsenal was made of 36 pounds of steel rod 1.02 inches in diameter. The outside diameter of the coil was 4.30 inches. A load of 11,000 pounds changed the length of this spring from 20.63 inches to 16.67 inches. After the load was removed the spring returned to its original length to within 0.02 inch. Find the energy per cubic inch and the energy per pound. *Ans.* 50.4 foot-pounds per pound.

4. In Problem 3 what was the maximum shearing stress due to torsion?

Ans. 86,580 pounds per square inch.

Miscellaneous Problems

1. Two 5-inch by 3-inch by ⅜-inch angles are placed with the backs of the 5-inch legs against the opposite sides of a ¾-inch plate, and all three pieces are fastened together by a single row of 1-inch rivets lengthwise the angles. The allowable tensile stress is 16,000 pounds per square inch, the allowable shearing stress in the rivets is 12,000 pounds per square inch, and the allowable bearing stress between the rivets and the plates is 24,000 pounds per square inch. How many rivets are required?

2. A shaft coupling is made of two disks which are connected together by ½-inch bolts arranged in the circumference of a circle 12 inches in diameter. The shaft is 4 inches in diameter and each disk is attached to a hub which is 6 inches in diameter. What must be the thickness of the

disk if the maximum shearing stress in it is the same as the maximum in the shaft? How many bolts are needed?

3. An 8-inch shaft is united to a fly-wheel by means of two keys, each 1 inch wide. The allowable shearing stress in the keys is 50 per cent. greater than the allowable shearing stress in the shaft. The allowable bearing stress is twice the allowable shearing stress in the shaft. Find the length of the hub and the thickness of each key.

CHAPTER VI

BEAMS

48. Definition of a Beam.—A beam is a rigid body which is subjected to parallel transverse forces. Figure 55 is the front view of a horizontal beam which is firmly held at the right end and carries a load P at the left end. The wall in which the beam is fixed exerts an upward pressure R_1 and a downward pressure R_2. These two *reactions* and the load P constitute the

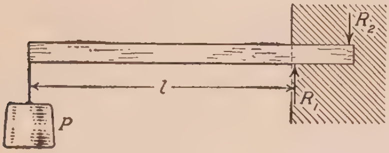


FIG. 55.

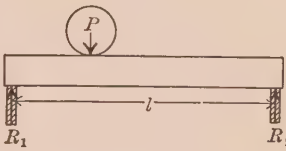


FIG. 56.—Beam supported at ends.

set of parallel transverse forces. Figure 56 shows a second beam which is supported at the ends and carries a concentrated load P between the supports. In addition to the *concentrated load* P, and the reactions R_1 and R_2, the weight of the beam itself furnishes another parallel transverse force. If the beam is uniform, this weight is a *uniformly* distributed load.

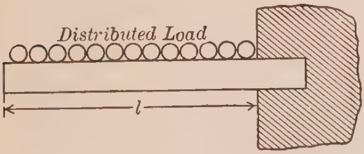


FIG. 57.—Cantilever.

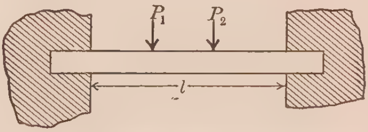


FIG. 58.—Beam fixed at both ends.

49. Kinds of Beams.—Beams may be classified according to the character of the support and the method of loading. Figure 57 represents a beam fixed at one end and free at the other. This kind of beam is called a *cantilever*. Figure 58 is a beam *fixed* at both ends. Figure 59 is *fixed* at the right end and *supported* at the left. Figure 60 is a beam which *overhangs* its supports.

A beam with three or more supports, as in Fig. 61, is a *continuous*, beam.

The figures show different methods of loading and some of the ways of representing the loads and reactions in diagrams and drawings.

Figures 55, 56, and 59, show a single concentrated load on each beam. In Fig. 58, there are two concentrated loads. In Fig. 57, there is a uniformly distributed load over the entire length. In Fig. 60, there is a uniformly distributed load over half the length, and a concentrated load at the right end. In Fig. 61, there is a distributed load over part of the left half and another load of greater weight per unit length over the right half.

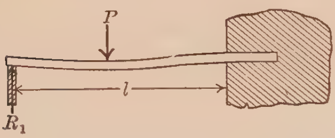


Fig. 59.—Beam fixed at one end and supported at the other.

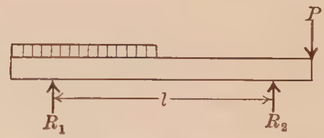


Fig. 60.—Beam overhanging its supports.

A beam is not necessarily horizontal. A vertical fence post subjected to the horizontal force of the wind or the weight of a gate is a cantilever beam. A post at the end of a line of wire fence is a vertical beam which is supported (horizontally) at the top and partially fixed at the bottom, and has a concentrated load at each wire.

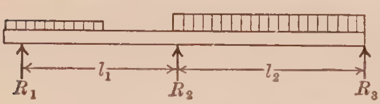


Fig. 61.—Continuous beam.

50. Reactions at Supports.—The calculation of the reactions at the supports of a beam is a problem of the equilibrium of non-concurrent, coplanar forces. The general problem of non-concurrent, coplanar forces has three unknown quantities, and requires three independent equations. When all the forces are parallel, as in most cases of beams, there are only two unknowns, and only two independent equations are required. One of these equations *must* be a *moment* equation; the other *may* be either a *moment* equation or a *resolution* equation. It is best to solve by two moment equations, and then check by a resolution equation. In order to eliminate one unknown, the origin of moments for the first equation, at least, should lie on one of the unknown reactions.

Example I

A uniform beam 12 feet long, weighing 60 pounds, is supported at the ends and carries a load of 72 pounds 4 feet from the left support (Fig. 62). Find the reactions at each support. Remembering that the center of gravity of a uniform beam is at the middle of its length, take moments about a horizontal line perpendicular to the beam through the right support.

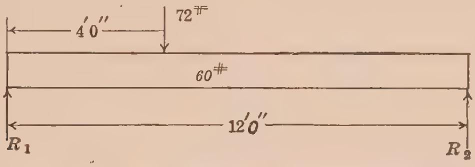


FIG. 62.—Beam supported at ends.

Load in pounds		Arm in feet		Moment in foot-pounds
60	×	6	=	360
72	×	8	=	576
R_1	×	12	=	936
		R_1	=	78 pounds.

Taking moments about the left support:

$$60 \times 6 = 360$$
$$72 \times 4 = 288$$
$$R_2 \times 12 = 648$$
$$R_2 = 54 \text{ pounds.}$$

Check by vertical resolutions.

Loads	Reactions
60	54
72	78
132	132

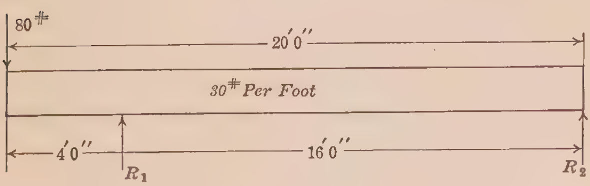


FIG. 63.—Beam overhanging left support.

Example II

A beam 20 feet long, weighing 30 pounds per foot is supported at the right end and 4 feet from the left end and carries a load of 80 pounds at the left end (Fig. 63). Find the reactions, and check.

The total weight of the beam is $30 \times 20 = 600$ pounds. Taking moments about an axis through the right support.

$$\begin{array}{rl} 600 \times 10 = & 6{,}000 \\ 80 \times 20 = & 1{,}600 \\ \hline R_1 \times 16 = & 7{,}600 \\ R_1 = & 475 \text{ pounds.} \end{array}$$

Taking moments about an axis through the left support,

$$\begin{array}{rl} 600 \times \quad 6 = & 3{,}600 \\ 80 \times -4 = & -320 \\ \hline R_2 \times \quad 16 = & 3{,}280 \\ R_2 = & 205 \text{ pounds.} \end{array}$$

It will be noticed that in the first part of each example counterclockwise moment is written positively and in the second part clockwise is written positively. This is done for convenience to avoid negative signs as much as possible. It makes no difference which sign is given to a moment expression, provided the same convention is retained throughout one equation. When the moments are not all of the same sign, it is convenient to take as positive the rotation which has the greatest number of terms. The direction of a moment should always be determined by noting which way it tends to rotate about the axis of moments rather than by observing the mathematical sign of the forces and the arms.

In the second example, the moment of the left 4 feet of the beam is counterclockwise about the left support while that of the remaining 6 feet is clockwise. Some students would write these two portions separately taking 120 pounds with a moment arm of 2 feet and 480 pounds with a moment arm of 8 feet. The method used in the illustrative example, where the whole weight is treated as concentrated at its center of gravity 6 feet from its right support, is shorter. Again, some would write these moments in the form of an equation, the first part of the second example being

$$600 \times 10 + 80 \times 20 = 16R_1.$$

This is sometimes convenient when there are factors which can be cancelled, but generally it is better to arrange the work as shown in the example. Where there are a large number of terms, several of which are negative, it is advisable to put the

positive moments in one column and the negative moments in another.

Problems

1. A uniform beam, 24 feet long, weighing 60 pounds per foot, is supported 2 feet from the left end and 6 feet from the right end, and carries a load of 480 pounds on the left end and a load of 320 pounds 8 feet from the left end. Find the reactions and check.
2. A horizontal beam, which is 22 feet long and weighs 792 pounds, is supported at the left end and 4 feet from the right end. It carries a load of 360 pounds 6 feet from the left end and a load of 216 pounds on the right end. Find the reactions at the supports by two moment equations and check by a resolution equation. Check also by moments about the right end.
3. A beam, 24 feet long, is supported at the left end and 4 feet from the right end. The beam weighs 60 pounds per foot and carries a load of 240 pounds. The left reaction is 648 pounds. Find the location of the 240-pound load and the reaction of the right support. Check.
4. A uniform beam is 18 feet long and weighs 288 pounds. It is supported at the left end and 2 feet from the right end, and carries a single concentrated load. The reaction of the left support is 262 pounds and the reaction at the right support is 346 pounds. Find the magnitude and position of the load and check.
5. A beam 4 feet long, weighing 60 pounds, with its center of gravity at the middle, is hinged at the lower right corner (Fig. 64) and held horizontal by a horizontal pull 8 inches above the hinge. Find this horizontal pull (H), the horizontal component of the hinge reaction (C), and the vertical component of the hinge reaction (V).

   Ans. H, 180 pounds; C, 180 pounds; V, 60 pounds.
6. The beam of Fig. 64 is hinged at the lower right corner and supported by a rope which is attached to a point 8 inches above the hinge. This rope makes an angle of 10 degrees above the horizontal toward the right. Find the tension in the rope and the components of the hinge reaction. Solve also graphically.
7. What should be the direction of the rope of Problem 6 in order that the vertical reaction at the hinge may be zero? Solve geometrically.

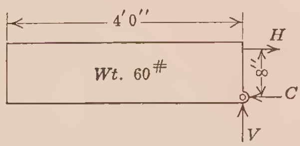


FIG. 64.—Beam supported by horizontal couple.

51. Shear in a Beam.—Figure 65 shows a cantilever beam which is fixed at the right end and carries a concentrated load P near the left end. A section EFG across this beam separates the left portion as a free body. Figure 69 shows the front elevation of the same beam. The load P is at a distance a from the left end, the section EFG is at a distance x from the left end, and the weight per unit length is w. When the portion of the beam to the left

of the section *EFG* is considered as a free body, the external forces are the load *P* and the weight of the portion, which is wx. The portion of the beam is kept in equilibrium by the internal forces which the right portion exerts on the left across the plane *EFG*.

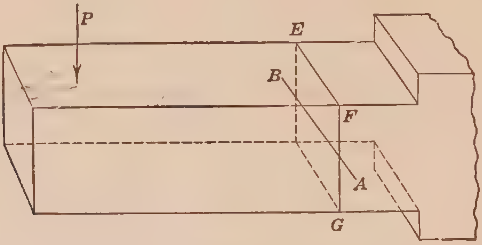


FIG. 65.—Section of cantilever.

Figure 66 shows the beam actually cut in two at the section *FG*. A cylinder, with its axis horizontal and perpendicular to the length of the beam, separates the two portions near the bottom, and a short horizontal chain connects them near the top.

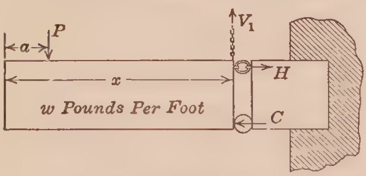


FIG. 66.—Cantilever shear and tension.

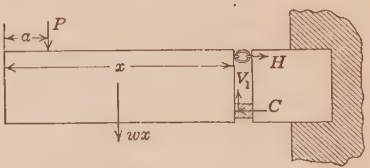


FIG. 67.—Cantilever shear resisted by friction.

A vertical chain is attached to the right end of the left portion of the beam.

The tension in each chain and the compression in the cylinder are calculated as a problem of the equilibrium of non-concurrent, coplanar forces. By a vertical resolution, the tension in the vertical chain is shown to be equal to the sum of the two vertical loads. This may easily be verified if the chain is supported by a spring balance. The horizontal resolution shows that the pull *H* of the horizontal chain is equal to the horizontal push *C* of the cylinder.

In Fig. 67 the cylinder is replaced by a rectangular block. If the coefficient of friction is sufficiently large, the friction will exert a vertical force equal to the weight of the portion, and the vertical chain may be removed. This vertical force is transmitted across the rectangular block as vertical shear.

Figure 68 shows the same beam with the pieces glued together. The lower part of the glue is in compression, the upper part is in tension, and all of it is in shear.

Figure 69 represents a similar beam which has not been cut. The material in it at any section is under the same stresses as the glue of Fig. 68.

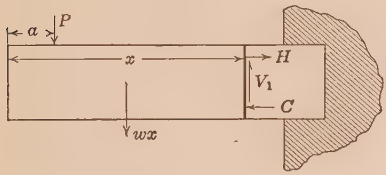


FIG. 68.—Resisting shear and moment at glued section.

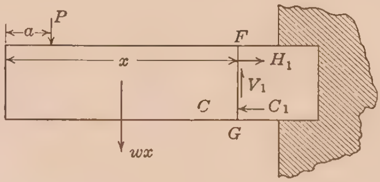


FIG. 69.—Shear and moment at section.

The vertical shear, V_1 of Fig. 69, is called the *resisting shear*. The resultant of all the forces parallel to the section which act on the portion of the beam on either side of the section is called the *external shear*. In a horizontal beam the external shear (for a section at right angles to the beam) is vertical and is called the *total vertical shear*. In formulas total vertical shear is represented by V. The resisting shear on one side of any section is equal and opposite to the external shear acting on the portion of the beam on the other side of the section. In Fig. 68 the external shear on the portion of the beam to the left of the section is $P + wx$ acting downward and is equal to the resisting shear with which the portion to the right of the section acts on the glue. Since the entire beam is in equilibrium under the action of the external forces, the external shear on the portion to the right of the section must be equal and opposite to the shear on the left portion. In like manner, the portion to the left of the section exerts a shear equal and opposite to V_1 upon the portion to the right.

The *magnitude* of the vertical shear may be determined from the vertical resolution of all the external forces which act on either the left or the right portion of the beam. In Figs. 68 and 69 it is convenient to use the left portion, since the *external* forces on this portion are given. In a cantilever *fixed* at the *left end* it would be better to consider the portion to the right of the section as the free body in equilibrium.

The *sign* of the vertical shear is determined from the resultant of the forces which act on the portion to the *left* of the section. The vertical shear is positive when the resultant of the vertical

forces to the left of the section is upward. It is positive when the *internal* shear acting from the left portion upon the right portion at the section is upward. In Figs. 68 and 69, the vertical shear is negative.

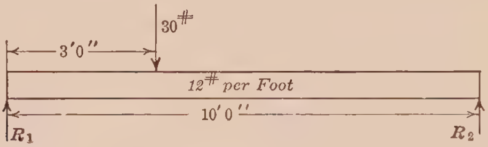


FIG. 70.—Beam supported at ends.

Example

A uniform horizontal beam, 10 feet long, weighing 12 pounds per foot, is supported at the ends and carries a load of 30 pounds 3 feet from the left end (Fig. 70). Find the total vertical shear at a section 2 feet from the left end and at a section 4 feet from the left end.

The reactions are 81 pounds at the left support and 69 pounds at the right support. Using the left portion as the free body in equilibrium, at 2 feet from the left end,

$$V_2 = 81 - 2 \times 12 = 57 \text{ pounds.}$$

At 4 feet from the left end,

$$V_4 = 81 - 4 \times 12 - 30 = 3 \text{ pounds.}$$

Using the right portion as the free body,

$$-V_4 = 69 - 6 \times 12 = -3 \text{ pounds.}$$

Just before the load of 30 pounds is reached, the shear is

$$V_{3-} = V_2 - 12 = 57 - 12 = 45 \text{ pounds.}$$

Just after the load is passed, the shear is

$$V_{3+} = 45 - 30 = 15 \text{ pounds.}$$

Problems

1. Calculate the shear at every foot for the beam of Example I of Article 50. Find the shear at 6 feet and at 8 feet from the left end by means of the portion to the right of the section. Why do you change the sign?
2. Find the shear at every foot for the beam of Example II of Article 50.

52. Bending Moment and Resisting Moment.—To calculate the compressive and tensile forces at any section one moment equation must be written. The moments are calculated with respect to an axis in the plane of the section *FG* (Figs. 71 and 69) perpendicular to the external forces. (In Fig. 69 this axis is perpendicular to the plane of the paper.) Considering the portion of the beam to the left of the section (Fig. 69) as a

free body, the *moment* of the *resisting shear* is zero (since its line of action passes through the axis of moments) and the moment of the *external forces* to the left of the section must be equal and opposite to the moment of the forces, H_1 and C_1, which act across the section from the right portion. It is advisable to take the axis through the center of gravity of the cross-section. This axis is the line AB of Fig. 71. The moment

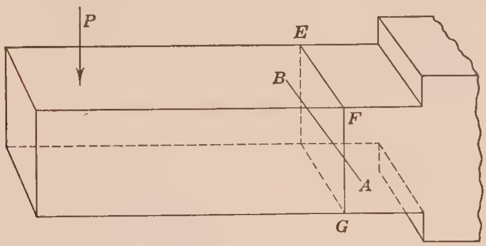


FIG. 71.—Moment of cantilever section.

about this axis of all the external forces which act on the free portion of the beam is called the *external moment*, or the *bending moment*. In Fig. 71, the external moment is the moment of the force P and of the weight of the left portion of the beam with respect to the axis AB. The moment of the internal forces with respect to the same axis is called the *resisting moment*. In Fig. 69, the resisting moment is the moment of the horizontal forces H_1 and C_1 about an axis through GF perpendicular to the plane of the paper.

Since the resisting moment and the external moment are always equal, it is customary to speak of the *moment* at the section.

When the external reactions and loads are all perpendicular to the length of the beam, and, consequently, are parallel to the section, the moment arms of all these forces may be measured from the plane of the section, instead of from some particular axis in this plane. When the external forces are perpendicular to the length, the internal forces are equal and opposite and, consequently, form a couple. The moment of a couple is the same for every axis perpendicular to the plane of the forces. For these reasons, the moment may be calculated about any axis in the plane of the section, provided the axis is perpendicular to the plane of the applied forces. It is customary, there-

fore, to speak of the "moment at a section" without reference to any particular line in that section.

The bending moment is considered positive at any section when the *portion* to the *left* of the *section* tends to turn the *portion* to the *right* in a *clockwise* direction, or when the *resisting moment* from *right* to *left* is *counterclockwise*. The bending moment is negative in Figs. 69 and 71 and is positive in Fig. 70.

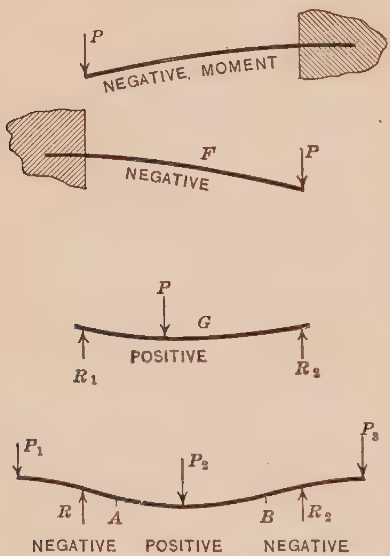


FIG. 72.—Positive and negative moment.

In the cantilever fixed at the left end in Fig. 72, the moment at all sections is negative. The portion to the right of F in this cantilever tends to turn the portion to the left of F in a clockwise direction; consequently, the portion to the left of F tends to turn the portion to the right of F in the opposite, or counterclockwise direction. The moment, therefore, is negative. In the beam which is supported near the ends, the portion to the left of P is evidently bending the remainder in a clockwise direction. Between the load and the right support, the moment of the left reaction is positive while that of the load is negative. With the supports at the ends, as in Fig. 70, the moment of the reaction is everywhere greater than the moment of the load. The moment is, therefore, positive throughout the entire beam. This may be proved if the portion to the right of G is taken as the free body. The right reaction R_2 turns this portion counterclockwise. The moment of the left portion must, therefore, be clockwise or positive.

The beam which overhangs the supports has negative moment from the left end to a point A, which is some distance to the right of the left support, and from the point B to the right end. Between A and B the moment is positive.

From the bent beams of Fig. 72 it is seen that with negative moment the beam is convex upward and that the center of curva-

ture is downward. With the positive moment the beam is concave upward. The most convenient method of determining the sign of the moment in many cases is by means of the center of curvature. If the center of curvature is on the positive side (above the beam) the moment is positive; if the center of curvature is on the negative side of the beam the moment is negative.

When a beam is not horizontal, as in Fig. 73, the X axis is taken parallel to the direction of its length and the Y axis at right angles to the direction of the X axis in the counterclockwise direction from that axis. The moment is positive when the ordinate of the center of curvature is positive, and negative when the ordinate of the center of curvature is negative. In Fig. 73, the moment is positive from A to B and negative from B to C.

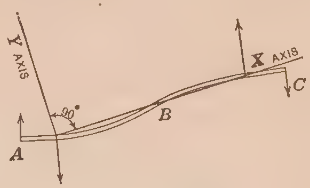


FIG. 73.—Inclined beam.

In formulas and equations, the bending moment is represented by M.

Example

A uniform horizontal beam 10 feet long, weighing 24 pounds per foot, is supported at the ends and carries a load of 60 pounds 3 feet from the left end. Find the moment and shear at 2 feet from the left end and at 5 feet from the left end.

The left reaction is 162 pounds and the right reaction is 138 pounds. At 2 feet from the left end the moment is that of the left reaction with an arm of 2 feet turning clockwise, and the weight of 2 feet of beam with a moment arm of 1 foot turning counterclockwise.

Force		Arm		Moment
162	×	2	=	324
−48	×	1	=	−48

$V = 114$ pounds $\qquad M = 276$ foot-pounds.

At 5 feet from the left end,

Force in pounds		Arm in feet	Moments in foot-pounds	
162		5.0	810	
	−120	2.5		−300
	− 60	2.0		−120
	−180			−420

$V_5 = -18$ pounds. $\qquad M_5 = 390$ foot-pounds.

The point B of Fig. 73 is called a point of *inflection*, or a point of *counterflexure*. The moment at this point is zero and changes sign.

Problems

(In all problems of this kind, make a sketch of the beam, put in all the dimensions and loads. After the reactions have been calculated, put these on the sketch, preferably in parentheses to show that they are not part of the original data. Check every step.)

1. A beam 12 feet long, weighing 60 pounds per foot, is supported at the ends and carries a load of 120 pounds 3 feet from the left end. Find the moment and shear at 2 feet from the left end, 3 feet from the left end, and 6 feet from the left end.

Ans. $V_2 = 330$ pounds; $M_2 = 780$ foot-pounds; $V_6 = -30$ pounds; $M_6 = 1{,}260$ foot-pounds.

2. A beam 12 feet long, weighing 20 pounds per foot, is supported at the ends. Find the moment at 3 feet from the left end and at the middle.

Ans. $M_6 = 360$ foot-pounds.

3. In Problem 1, find the moment at 6 feet from the left end by means the portion to the right of the section as the free body.

4. A beam of length l is supported at the ends and carries a load P at the middle. Find the moment at the middle, at one-fourth the length from the left end, and at three-fourths the length from the left end.

Ans. At the middle, $M = \frac{Pl}{4}$.

5. A beam of length l is supported at the ends and carries a uniformly distributed load of w per unit length. Find the moment at the middle, at one-fourth the length from the left end, and at three-fourths the length from the left end.

Ans. At the middle, $M = \frac{wl^2}{8}$.

6. Solve Problem 2 by substituting in the answer of Problem 5.

7. A beam 15 feet long weighs 40 pounds per foot. It is supported at the right end and at 3 feet from the left end, and carries a load of 100 pounds on the left end. Find the shear and the moment at 2 feet, 3 feet, 4 feet, 5 feet, and 6 feet, from the left end.

8. A beam 20 feet long, weighing 80 pounds per foot, is supported at the left end and 4 feet from the right end, and carries a load on the right end. Find this load if the moment at 3 feet from the left end is 300 foot-pounds.

9. Find the moment at the fixed end of a cantilever of length l which is due to a load P at the free end. Also find the moment which is due to a uniform load W. Find the shear in each case at the fixed end.

Ans. Moments, $-Pl$, $-\frac{Wl}{2}$; shear, $-P$, $-W$.

10. If an observer should pass from one side of a beam to the other (from front to rear), show that the sign of the shear as viewed from the new position will be reversed but the sign of the moment will not be changed.

53. Shear Diagrams.—It is often convenient to represent the total shear at all sections of a beam by means of a diagram. Figure 74 is the *shear diagram* for a uniform horizontal beam, which is 12 feet long, weighs 20 pounds per foot, and is supported at the ends. Each end reaction is 120 pounds. At any section of the beam, the shear is the algebraic sum of the left reaction upward and the weight of the portion of the beam

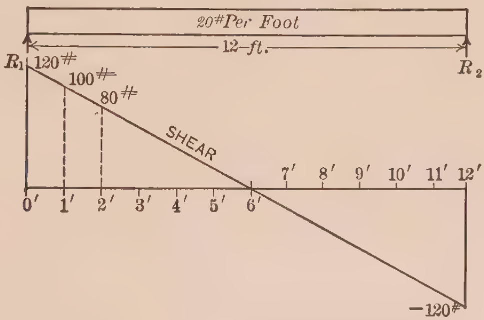


FIG. 74.—Shear diagram for distributed load.

between the left end and the section acting downward. Infinitely near the left support, the weight of the portion of beam to the left is negligible. The shear, therefore, is the left reaction of 120 pounds. Infinitely near the right support, the shear is the reaction of 120 pounds minus the weight of practically all the beam, which is 240 pounds. The shear is, therefore, minus 120 pounds. At one foot from the left end, the shear is

$$V_1 = 120 - 20 = 100 \text{ pounds.}$$

At 2 feet from the left end, the shear is 80 pounds. If the shear is calculated for every foot, and a line passed through each point, this line is found to be a straight line. The shear diagram may be constructed by constructing the points at the ends and joining these with a straight line. At x feet from the left support the shear is

$$V_x = 120 - 20x, \tag{1}$$

which is evidently the equation of a straight line.

Figure 75 is the shear diagram for a beam which is 10 feet long, weighs 60 pounds per foot, is supported at the ends, and carries a load of 200 pounds 3 feet from the left end. By moments about the right support, the left reaction R_1 is found to be 440 pounds.

By moments about the left support, the right reaction is found to be 360 pounds. The sum of these reactions is 800 pounds, which checks the total load.

The shear is 440 pounds infinitely near the left support. It drops 180 pounds in the first 3 feet and is 260 pounds infinitely close to the left of the load of 200 pounds. Under this load, the shear diagram drops vertically 200 pounds. The shear infinitely

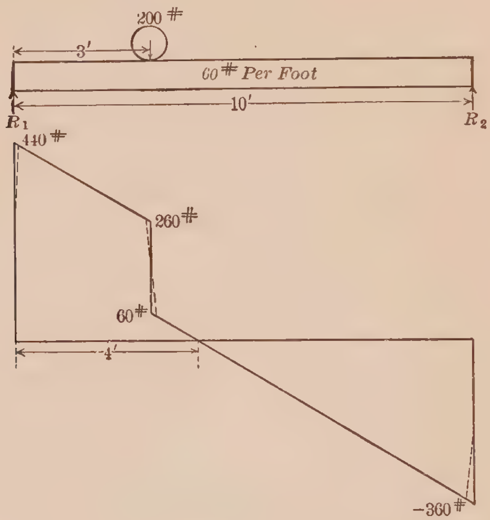


FIG. 75.—Concentrated and distributed loads.

close to the right of the 200-pound load is 60 pounds. Beyond the concentrated load, the shear drops at the rate of 60 pounds per foot for the remaining 7 feet. It is minus 360 pounds infinitely close to the left side of the right support. The right reaction of 360 pounds raises the diagram to the initial line. The diagram crosses the initial line, or zero ordinate, 1 foot to the right of the concentrated load, which is 4 feet from the left support.

The shear diagram of Fig. 75 is a vertical straight line at each support and at the concentrated load, and the discussion refers to points infinitely near the supports or the load. This method of treatment assumes that the loads and reactions act on mathematical lines. In reality, the *surface* of contact is a band of some width extending across the beam, and the actual shear diagram is something like that represented by the broken curved lines.

In Fig. 75, the shear decreases 60 pounds for every foot. With the origin of coördinates at the left end, the equation of shear for the first 3 feet is

$$V = 440 - 60x. \tag{2}$$

For the remainder of the beam,

$$V = 440 - 200 - 60x = 240 - 60x. \tag{3}$$

If the origin of coördinates is taken at the concentrated load, the equation of the shear for the beam to the right of that load is

$$V = 60 - 60x. \tag{4}$$

It is frequently desirable to locate the point at which the shear is zero. This may be done by means of the equation of the shear diagram. If Equation (2) were used, the result would not be the point desired, since this equation applies only to the part of the beam to the left of the load. Equation (3) gives the point 4 feet from the left end and Equation (4) gives it 1 foot from the load of 200 pounds.

Generally, it is better not to think of the equation of the line. At 3 feet the shear drops to 60 pounds. It continues to drop at the rate of 60 pounds per foot, in what distance will it become zero?

Problems

1. A horizontal beam is 12 feet long, weighs 50 pounds per foot, and is supported at the right end and 2 feet from the left end. It carries a load of 200 pounds 2 feet from the right end. Construct the shear diagram with 1 inch in the horizontal direction equal to 2 feet of length and 1 inch in the vertical direction equal to 100 pounds of shear. Calculate the position of zero shear and compare with the diagram.
2. A horizontal beam is 12 feet long, and weighs 20 pounds per foot. It is supported 4 feet from the left end and held down at the left end. It carries a load of 60 pounds per foot from the second support to the right end. Construct the shear diagram with abscissas as in Problem 1, and with ordinates of 1 inch equal to a shear of 200 pounds.
3. A beam 20 feet long carries a load of 200 pounds per foot, including its own weight. It is supported at the ends and carries a load of 800 pounds 5 feet from the left end and a load of 400 pounds 2 feet from the right end, Draw the shear diagram to a suitable scale.
4. A beam of length l is supported at the ends and carries a load P at the middle. If the weight of the beam is neglected, draw the shear diagram.
5. A beam of length l is supported at the ends and carries a load $P/2$ at a distance a from the left end and an equal load at a distance a from the

right end. Construct the shear diagram, neglecting the weight of the beam. Where is the shear zero?

6. A beam of length l is supported at the ends and carries a load P at a distance a from the left end and a load Q at a distance b from the right end. The loads and distances are such that $Pa = Qb$. Neglecting the weight of the beam, find the reactions and construct the diagram.

Shear diagrams are usually made up of straight lines. These lines are horizontal from one load to the other when the loads are concentrated and the weight of the beam is neglected. With uniformly distributed loads, the lines slope downward from left to right. (With distributed loads pushing up, as in the bottom

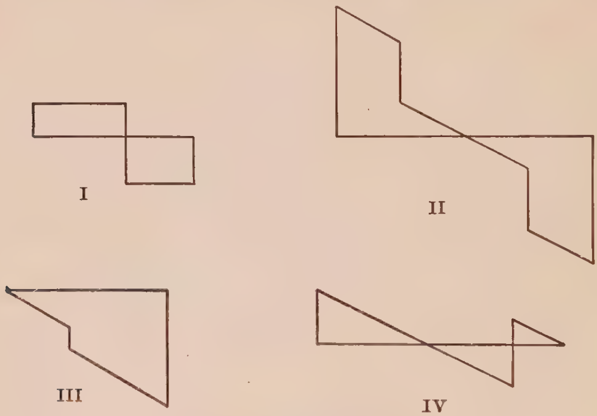


FIG. 76.—Shear diagrams.

of a boat subjected to water pressure, the lines slope upward.) Where loads are distributed not uniformly, as in the case of the water pressure on a vertical dam, the shear diagram is curved.

The student should become sufficiently familiar with the simpler shear diagrams to be able to recognize the character of the loading at a glance.

Problem

7. Describe the loading and the character of support which gives each of the shear diagrams of Fig. 76.

54. Moment Diagrams.—Moment diagrams are constructed in the same way as shear diagrams. The abscissas represent horizontal distances in the beam, and the ordinates represent the external moments. In this book, positive moment is drawn upward. Some writers, however, prefer to draw the moments opposite to the method here used.

Since shear diagrams usually consist of straight lines, they are easy to construct. Moment diagrams are curved, except when all the loads are concentrated.

Figure 77 shows the shear and moment diagram for a beam supported at the ends and carrying a load P at the middle. The

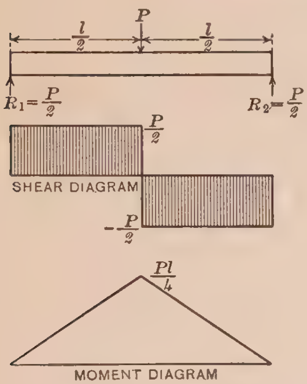


FIG. 77.—Singly concentrated load.

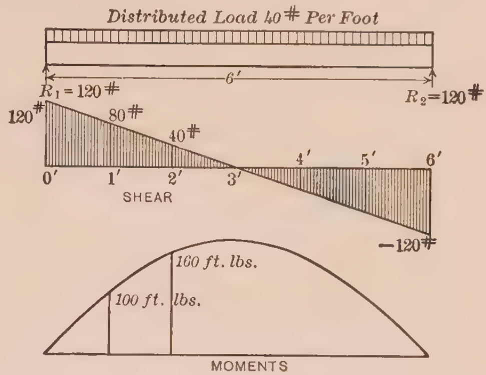


FIG. 78.—Uniformly distributed load.

weight of the beam is neglected. The end reactions are $\frac{P}{2}$. The moment at any section at a distance x from the left end is $\frac{Px}{2}$, provided x is not greater than one-half of the length. Under the load the moment is $\frac{Pl}{4}$. The moment diagram for the left half of the beam is a straight line through the points (0, 0) $\left(\frac{l}{2}, \frac{Pl}{4}\right)$. Beyond the concentrated load, the moment caused by the reaction at the left end is diminished by the moment caused by the load at the middle. At a distance x from the left end, when x is greater than $\frac{l}{2}$,

$$\text{Moment} = \frac{Px}{2} - P\left(x - \frac{l}{2}\right) = \frac{Pl}{2} - \frac{Px}{2} = \frac{P}{2}(l - x).$$

This also is a straight line. The last of the expressions for the moment may be obtained directly by using the portion to the right of the section as the free body. The right reaction is $\frac{P}{2}$ and its moment arm is $l - x$.

Figure 78 gives the shear and moment diagrams for a beam which is supported at the ends and carries a uniformly distributed load. The moment diagram is a parabola with the vertex at the top.

Example

A beam 12 feet long weighs 20 pounds per foot. It is supported at the left end and 2 feet from the right end, and carries a load of 100 pounds 2 feet from the left end and a load of 80 pounds at the right end. Construct the shear and moment diagrams to the scale of 1 inch horizontal equals 2 feet of length, and 1 inch vertical equals 100 pounds shear and 100 foot-pounds moment.

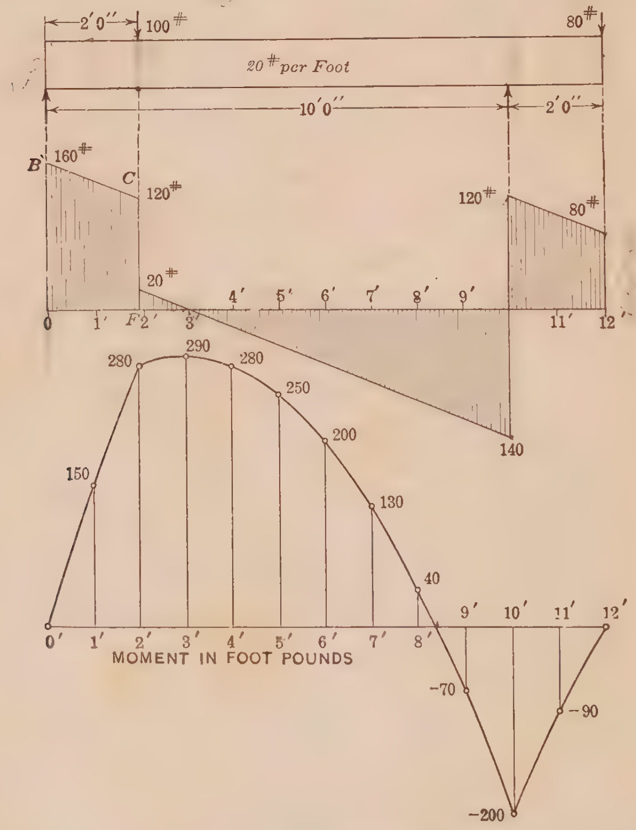


FIG. 79.—Shear and moment diagrams.

Figure 79 shows the curves for this example. Under the load of 100 pounds, the shear diagram drops from 120 pounds to 20 pounds and the moment diagram has an abrupt change of

curvature. At the right support, the shear diagram rises vertically and the moment diagram has a still more abrupt change of curvature and direction. The shear diagram crosses the X axis 3 feet from the left end, at which point the moment is a maximum. The shear again crosses the X axis at the right support, where the moment is a minimum.

Problems

1. A beam 10 feet long is supported at the ends and carries a distributed load of 600 pounds per foot, which includes its own weight. A concentrated load of 2,000 pounds is applied 3 feet from the right end. Construct the shear and moment diagrams to the scale of 1 inch horizontal equals 2 feet and 1 inch vertical equals 1,000 pounds shear and 2,000 foot pounds moment. Find the position of zero shear algebraically, and compare with the diagram.

2. Construct the shear and moment diagrams for a beam of length l, which is supported at the ends and carries a load P at six-tenths the length from the left end. Neglect the weight of the beam. Compare with Fig. 77.

3. Construct the moment diagram for Problem 5 of Article 53.

4. Write the equation of the moment curve of Fig. 78.

5. Write the equation of the moment curve for a beam of length l, which is supported at the ends and carries a uniformly distributed load of w pounds per unit length.

$$\textit{Ans.}\ M = \frac{wlx}{2} - \frac{wx^2}{2} = \frac{wx}{2}(l - x).$$

6. In Problem 5, how far from the ends is the moment one-half as great as the moment at the middle?

7. Write the equation of the moment curve of the example, Fig. 79, in terms of the distance from the left end.

$$\textit{Ans.}\ \begin{cases} M = 160x - 10x^2, \text{ from 0 to 2 feet.} \\ M = 60x + 200 - 10x^2, \text{ from 2 feet to 10 feet.} \\ M = 320x - 2{,}400 - 10x^2, \text{ from 10 feet to 12 feet.} \end{cases}$$

8. Find the moment at 2 feet from the left end by both the first equation and second equation of Problem 7, and find the moment at the right support by the second equation and third equation.

9. Solve the second equation of Problem 7 for the position of which the moment is zero. There are two solutions of the algebraic equation. Which one should be taken? Why? What is this point called?

10. Apply the mathematical condition of maximum and minimum to the second equation of Problem 7 to find the position of maximum moment.

55. The General Moment Equation.—In the examples which have been given, the origin of coördinates has been taken at the left end of the beam. But it is often desirable to be able to write the moment equation with any point as the origin. Figure 80 represents a beam of indefinite extent with the origin

of coördinates on a vertical line through O. To the right of the origin, at distance a_1, a_2, etc., there are concentrated loads P_1, P_2. etc. There is also a uniformly distributed load of w per unit length. There may be any number of vertical loads and reac-

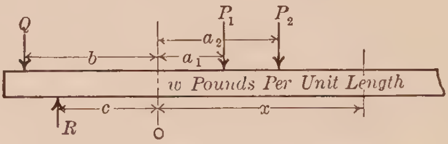


FIG. 80.—General case of loading.

tions to the left of the origin, but all the vertical loads may be replaced by their resultant Q at some definite distance b from the origin, and all the vertical reactions by a single reaction R at a distance c from the origin. Writing the moment with respect to a section at a distance x from the origin,

$$M = R(c + x) - Q(b + x) - P_1(x - a_1) - P_2(x - a_2) - \frac{wx^2}{2}; \quad (1)$$

$$M = Rc - Qb + (R - Q)x - P_1(x - a_1) - P_2(x - a_2) - \frac{wx^2}{2}. \quad (2)$$

$Rc - Qb$ is the moment at the origin, which may be represented by M_0, and $R - Q$ is the shear at the origin, which may be represented by V_0.

$$M = M_0 + V_0x - P_1(x - a_1) - P_2(x - a_2) - \frac{wx^2}{2}; \quad (3)$$

$$M = M_0 + V_0x - \Sigma P(x - a) - \frac{wx^2}{2}, \quad \text{Formula IX.}$$

in which $\Sigma P(x - a)$ represents the sum of the moments of all the concentrated loads between the origin and the section considered. When any point on a beam is taken as the origin of coördinates, the moment at any section at a distance x to the right of the origin is the moment at the origin, plus the shear at the origin multiplied by the distance of the section from the origin, plus the moment with respect to the section of each load and reaction between the origin and the section.

Example I

For the beam of Fig. 79, find the moment at 6 feet from the left end by means of the general moment equation, taking the origin of coördinates infinitely close to the right of the load of 100 pounds.

$$M_0 = 280 \text{ foot-pounds; } V_0 = 20 \text{ pounds.}$$
$$M = 280 + 20 \times 4 - 80 \times 2 = 200 \text{ foot-pounds.}$$

Problems

1. For the beam of Fig. 79, with the origin of coördinates as in the above example, find the moment at 8 feet, at 10 feet, at 11 feet, and at 12 feet from the left end.
2. Solve Problem 1 with the origin of coördinates 3 feet from the left end.

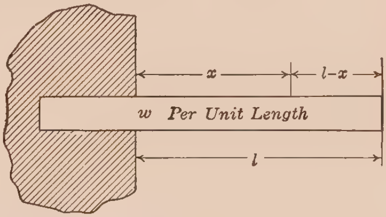


FIG. 81.—Cantilever fixed at left end.

Example II

A cantilever of length l and weight w per unit length is fixed at the left end. Take the origin of coördinates at the fixed end and derive the expression for the moment at a distance x from the origin by means of the general moment equation.

The entire beam to the right of the origin (Fig. 81) must be taken as the free body to find M_0 and V_0.

$$M_0 = -wl \times \frac{l}{2} = -\frac{wl^2}{2};$$
$$V_0 = wl.$$
$$M = -\frac{wl^2}{2} + wlx - \frac{wx^2}{2} = -\frac{w}{2}(l - x)^2.$$

The last form of the moment may be checked by means of the portion of the beam to the right of the section as the free body. The weight of this portion is $w(l - x)$ and its moment arm is $\frac{l - x}{2}$. This last method is the one which would generally be used for a problem of this kind. The general moment equation is applied chiefly to continuous beams, such as those in Chapter IX.

Problems

3. A beam 24 feet long, which weighs 60 pounds per foot, rests on three supports, spaced 10 feet apart, and overhangs the left support 4 feet. By methods of Chapter IX, the shear infinitely close to the right of the left support is found to be 285 pounds. Write the equation for the moment for sections between the first and second support. Find the moment at 5 feet from the left support and also over the second support.
4. In the beam of Problem 3, write the moment equation for the span between the second and third support with the origin of coördinates

infinitely close to the right of the left support. Represent the unknown reaction at the middle support by R_2. Calculate the moment over the right support by this equation and solve for R_2.

56. Relation of Moment and Shear.—Differentiate the general moment equation with respect to x.

$$M = M_0 + V_0x - P_1(x - a_1) - P_2(x - a_2) - \frac{wx^2}{2}. \tag{1}$$

$$\frac{dM}{dx} = V_0 - P_1 - P_2 - wx. \tag{2}$$

The right member of (2) is recognized as the shear at a distance x from the origin.

$$\frac{dM}{dx} = V. \qquad \text{Formula X.}$$

The derivative, with respect to the length, of the moment equation of a beam gives the shear in the beam.

In Fig. 79, there is an abrupt change in the slope of the moment curve at the concentrated load and at the second support. At the concentrated load, the shear changes from 120 pounds to 20 pounds and there is an equivalent relative change in the slope of the tangent to the moment curve. The shear at this point may be said to have any value between 120 pounds and 20 pounds. The derivative of the moment is not *single-valued* and Formula X does not hold. It does hold, however, infinitely close to this point on either side.

In reality, no load can be concentrated at a point or on a line extending across the beam. A so-called concentrated load is actually distributed over an area. If this distribution were known, the shear at any point would have a single value and Formula X would be found to be valid at all sections.

Since

$$V = \frac{dM}{dx},$$

$$\int V dx = \int dM;$$

$$\int V dx = M_2 - M_1. \qquad \text{Formula XI.}$$

The integral of Vdx between any two values of x gives the difference of the moments at the corresponding points.

Figure 82 is part of the shear diagram for a beam which is supported at the left end, weighs w pounds per foot, and carries a

load P at a distance a from the left end. The element of width dx at a distance x from the left of the diagram extends from the X axis to the line BC. The area of this element is Vdx. The integral of Vdx between the limits x_1 and x_2 represents the area

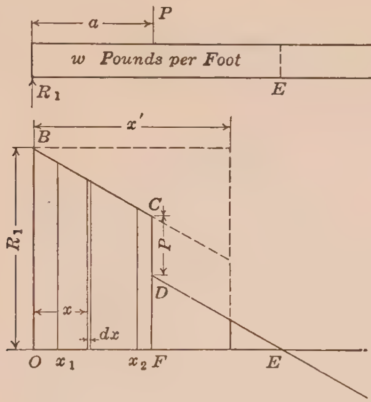


FIG. 82.—Relation of area of shear diagram to moment.

which is bounded by the shear diagram, the X axis, and the ordinates x_1 and x_2. Since $\int Vdx = M_2 - M_1$, *the area of the shear diagram between two points is the difference between the moments at these points.* When the shear is negative, the area is below the X axis, and, therefore, is negative.

At the ends of a beam, if all the loads and reactions are perpendicular to the length, the moment is zero. It follows, therefore, that the moment at any section is the entire area of the shear diagram from the end of the beam to the section.

Example

In Fig. 79, find the moment at 2 feet, 3 feet, 4 feet and 7 feet from the left end by means of the area of the shear diagram.

At 2 feet, the moment is the area of the trapezoid the base of which is 2 feet, and the altitude is 160 units on one side and 120 units on the other side.

$$M_2 = 0 + \frac{160 + 120}{2} \times 2 = 280 \text{ foot-pounds.}$$

At 3 feet the moment is the moment at 2 feet plus the area of the triangle 20 units high and 1 foot wide.

$$M_3 = 280 + 10 = 290 \text{ foot-pounds.}$$

At 4 feet the negative triangle is subtracted from the moment at 3 feet.

$$M_4 = 290 - 10 = 280 \text{ foot-pounds.}$$

At 7 feet, the negative triangle has a base of 4 feet and an altitude of 80 units.

$$M_7 = 290 - \frac{80 \times 4}{2} = 130 \text{ foot-pounds.}$$

Problems

1. In Fig. 79, find the moment at 5 feet, 9 feet, 10 feet and 12 feet from the left end by means of the area of the shear diagram, assuming that the moment at 3 feet is known.
2. In Problem 1 of Article 54, find the moment at 2 feet, 3 feet, 4 feet, 7 feet and 10 feet from the left end by means of the shear diagram.
3. A cantilever of length l is fixed at the right end and carries a load P at the left end. Calculate the moment at the right end by means of the shear diagram.
4. A cantilever of length l is fixed at the right end and carries a load of w per unit length, uniformly distributed. Find the moment at the right end by means of the area of the shear diagram.
5. A beam of length l is supported at the ends and carries a load P at the middle. Find the moment at one-fourth the length from the left end, and also at three-fourths the length from the left end by means of the shear diagram.
6. A beam of length l, which is supported at the ends, weighs w pounds per unit length. Find the moment at the middle and at a distance x from the left end by means of the area of the shear diagram.
7. Find the moment for any point between the loads for the beam of Problem 5 of Article 53.
8. Find the moment at any point on the beam of Problem 6 of Article 53 by means of the shear diagram. Draw the moment diagram.
9. In Fig. 82 write the expression for the moment at a distance x' from the left end, where x' is greater than a. and show what areas on the figure represent the terms of the result.

57. The Dangerous Section.—A section in a beam where the moment has a maximum numerical value is called a *dangerous section*. The mathematical condition for a maximum or minimum value of M is that the derivative with respect to the length shall be zero. But since $\frac{dM}{dx}$ is the shear, this means that there is a dangerous section at every point where the shear becomes zero In Fig. 79 the shear diagram crosses the X axis at 3 feet from the left end. This is one dangerous section.

The shear may pass through zero when the moment equation does not fulfill the mathematical condition that the slope of the tangent to the curve is zero. At the right support in Fig. 79, the slope of the moment curve changes abruptly from negative to positive. The negative moment at this point has the maxi-

mum numerical value. This is evident from the shear diagram. The shear changes from negative to positive at the support. A positive area must, therefore, be added to the negative moment after the support is passed.

When the loading of a beam is given, always find the dangerous sections by means of a sketch of the shear diagram. If the dangerous section does not come at a support or under a concentrated load, its exact position may be found algebraically.

Problems

1. Find the moment at the dangerous section of Problem 1 of Article 54.

2. A beam 16 feet long, weighing 120 pounds per foot, is supported 3 feet from the left end and 1 foot from the right end. It carries 300 pounds on the left end, 660 pounds on the right end, and 1,080 pounds 5 feet from the left end. Draw the shear diagram and locate each dangerous section. Find the moment at each dangerous section algebraically and check by the area of the shear diagram. Write the equation of moments for the portion between the left support and the load of 1,080 pounds and solve for the position of zero moment. The equation has two roots. Which one should be taken? Why? Write the equation and find the other position of zero moment.

3. A beam 12 feet long, weighing 60 pounds per foot, is supported at the ends, and carries a load of 240 pounds 3 feet from the left end and a load of 720 pounds 2 feet from the right end. Find the moment at the dangerous section.

4. Find the moment at the dangerous section for a cantilever 10 feet long, which carries a distributed load of 20 pounds per foot, including its own weight.

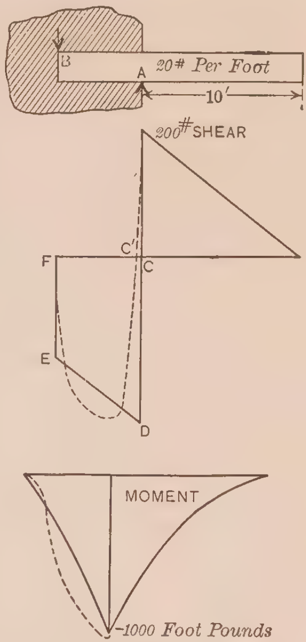


FIG. 83.—Cantilever fixed at left end.

Figure 83 shows the moment and shear diagrams for a cantilever. If there are no horizontal forces, the area of the shear diagram inside the wall must equal the area outside. The form of the diagram inside the wall is not known. If all the downward forces were concentrated at B, and all the upward force at A, the shear diagram would be the figure $CDEF$. Since the pressures are distributed, the shear diagram is that shown by the dotted

lines, and the dangerous section is at C', a little back of the face of the wall.

The actual moment diagram inside the wall is something like that shown by the dotted lines.

Miscellaneous Problems

1. A cantilever of length l is fixed at the right end and carries a uniformly distributed load of w per unit length. Draw the moment diagram. By integration find the area of the moment diagram.

Ans. Area of moment diagram $= -\frac{wl^3}{6}$.

2. A cantilever of length l is fixed at the right end and carries a load of w per unit length over a length a measured from the free end and no load over the remainder. Draw the shear diagram. Draw the moment diagram. Find the moment at the fixed end and check by the area of the shear diagram.

3. Solve Problem 2 if a is one-half the length.

Ans. Moment at the wall $= -\frac{3wl^2}{8}$.

4. A beam of length l is supported at the ends and carries a load P at the middle. Draw the positive moment triangle for the moment of the left reaction, and the negative triangle for the moment of the load. Combine these graphically and compare with Fig. 77.

5. A beam of length l is supported at the ends and carries a distributed load of w per unit length. Draw the positive triangle for the moment of the reaction and the negative parabola for the moment of the distributed load. Combine these diagrams graphically and compare with Fig. 78. Find the area of the entire moment diagram.

6. A horizontal cantilever of length l is fixed at the right end, and carries a load which increases uniformly from no load at the left end to a maximum at the right end, and is u pounds per unit length at unit distance from the left end. Draw the moment diagram and the shear diagram. Calculate the area of the moment diagram.

Ans. Shear $= -\frac{ux^2}{2}$. Moment $= -\frac{ux^3}{6}$. Area of moment diagram $= -\frac{ul^4}{24}$.

7. A beam of length l is supported at the ends and carries a load which increases uniformly from the left to the right end, and is u pounds per unit length at unit distance from the left end. Write the shear equation and draw the shear diagram. Write the moment equation and draw the moment diagram. Find the moment at the dangerous section in terms of u and also in terms of the total weight W.

Ans. $V = u\left(\frac{l^2}{6} - \frac{x^2}{2}\right)$. $M = \frac{ux}{6}(l^2 - x^2)$.

Maximum moment, at $x = \frac{l}{\sqrt{3}}$, is $\frac{2Wl}{9\sqrt{3}}$.

8. How does the maximum moment in a beam which is loaded as in Problem 7, compare with the maximum moment if the same load were uniformly distributed?
9. A beam of length l is supported at the ends and carries a load which increases uniformly from the middle to each end. Find the moment at the middle in terms of the total load W. Compare with Carnegie Pocket Companion.
10. A cantilever of length l is fixed at the right end and carries a distributed load over one-half the length from the free end and no load over the remainder. Write the moment equation for each half of the beam with the origin of coordinates at the free end. Draw the moment and shear diagrams.
11. A beam of length l is supported at the ends and carries a uniformly distributed load over the left half and no load over the remainder. Draw moment and shear diagrams. Find the moment at the dangerous section by means of the area of the shear diagram and also from the definition of moment.
12. A beam supported at the ends is 20 feet long. It carries 120 pounds per foot over its entire length, 600 pounds per foot additional over the 12 feet adjacent to the left support, 800 pounds 6 feet from the left end, and 400 pounds 5 feet from the right end. Draw the moment and shear diagrams. Find the moment at the dangerous section.
13. A beam, 25 feet long, is supported 3 feet from the left end and 2 feet from the right end. It carries 200 pounds per foot uniformly distributed, 600 pounds 1 foot from the left end, 800 pounds 7 feet from the left end, and 1200 pounds 6 feet from the right end. Draw the shear diagram and find the moment at the dangerous section.

CHAPTER VII

STRESSES IN BEAMS

58. Distribution of Stress.—At any section of a bent beam, there is tension across the part adjacent to the convex surface and compression across the part adjacent to the concave surface, and there is usually shear parallel to the section. The method of finding the total vertical shear has been given in Chapter VI. The method of determining the unit shearing stress will be given later in Chapter X. The problem of the total tension and compression and of the unit tensile and compressive stresses will now be considered.

If the external forces have no components parallel to the length of the beam, the resultant compressive stress across any section is equal to the resultant tensile stress, and these two forces form a couple, the moment of which is equal to the product of either force multiplied by the distance between them. This moment is equal and opposite to the bending moment.

To calculate these forces (H and C of Figs. 64 to 69) it is only necessary to know the bending moment and the distance between the forces. This distance is easily measured in Figs. 64 and 66. In Fig. 67 the compressive stress is distributed over the small block, and its law of distribution must be known in order to locate its resultant.

In Fig. 69, the tensile stress is distributed over the entire upper portion and the compressive stress is distributed over the entire lower portion. In order to find the moment arm of the couple, it is necessary, therefore, to know the law of distribution of these stresses.

The fibers on the convex side of a bent beam are elongated and those on the concave side are shortened. Between these there is a surface in which the fibers suffer no deformation in the direction of the length of the beam. This surface is called the *neutral surface* of the beam. The intersection of the neutral surface with any transverse section of the beam is called the *neutral axis* of that section.

It is customary to assume that the unit stress at any section varies directly as the distance from the neutral axis. The reasons for this assumption and the conditions under which it is valid, will be given in Article 63. Figure 84 represents graphically the variation of unit stress in a beam, the upper part of which is in tension.

Figure 84, I, shows the forces from left to right and also the forces from right to left. Figure 84, II, shows only the forces with which the portion of the beam to the right of the section acts

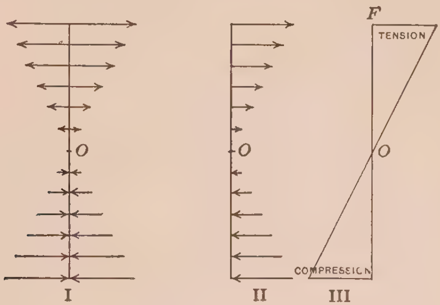


FIG. 84.—Stress variation in a beam.

on the portion to the left. It will be noticed that both sets of forces tend to turn the left portion clockwise about the neutral axis at O. Figure 84, III, shows a convenient method of drawing the diagram to show the magnitude of the unit stress at any distance from the neutral axis.

Since the unit stress varies as the distance from the neutral axis, it may be represented by two wedges cut from the beam by two planes which pass through the neutral axis. One of these planes should be normal to the length of the beam, and, therefore, represent the section considered, and the other may make any convenient angle. Figure 84, III, may be considered as representing two such planes. The volume of each wedge may be regarded as giving the total stress across its corresponding part of the section, and the distance between the center of gravity of the two wedges, measured parallel to the section, gives the moment arm of these total stresses.

59. Fiber Stress in a Beam of Rectangular Section.—Figure 85 shows the wedges representing the stress distribution of a rectangular beam section of breadth b and depth d. Since the total tension H is equal to the total compression C, the two wedges

which represent the total tension and compression must have equal volume. Since the slope and width of the wedges are the same and their volumes are equal, their heights must be equal, and the neutral axis BB' is at a distance $\frac{d}{2}$ from the top or bottom of the section. If S is the unit tensile stress in the outer fibers

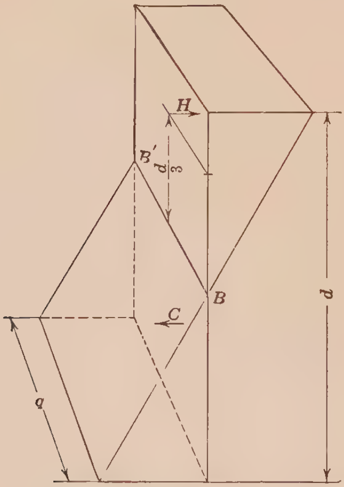


FIG. 85.—Solids representing stress in a rectangular beam.

at the top of the beam, $\frac{S}{2}$ is the average tensile stress over the upper half of the section. The total tension is the average stress multiplied by the area above the neutral axis.

$$H = \frac{S}{2} \times \frac{bd}{2} = \frac{Sbd}{4}.$$

Problems

1. A beam of rectangular section is 4 inches wide and 12 inches deep. The unit stress in the outer fibers at the convex surface is 1,000 pounds per square inch. What is the total tension?

   Ans. $500 \times 4 \times 6 = 12{,}000$ pounds.

2. The flange of a T-beam is 5 inches wide and 2 inches high. The thickness of the stem is 1 inch and the net height is 10 inches. The neutral axis is 4 inches from the top of the flange or 8 inches from the bottom of the stem. The tensile stress at the bottom of the stem is 1,200 pounds per square inch. What is the compressive stress at the top of the flange? What is the average compressive stress in the flange? What is the total

tension in the lower 8 inches of the stem? What is the total compression in the remainder of the section?

Ans. Total tension = 4,800 pounds.

3. A hollow box girder is 10 inches wide and 12 inches high outside, and is 6 inches wide and 8 inches high inside. The maximum unit stress in the top and bottom fibers is 1,200 pounds per square inch. Find the total tension in one half.

Ans. 26,400 pounds.

The line of application of the resultant tension H of Fig. 85 passes through the center of gravity of the wedge. Since the center of gravity of a triangle, and, consequently, of a triangular wedge is two-thirds the height from the vertex, the distance of H from the neutral axis is $\frac{2}{3} \times \frac{d}{2} = \frac{d}{3}$. In like manner the total compression C is located at a distance $\frac{d}{3}$ below the neutral axis. The total moment arm of the couple made up of the forces H and C is $\frac{2d}{3}$.

Problems

4. What is the moment about the neutral axis of the forces of Problem 1?

Ans. $12{,}000 \times 4 + 12{,}000 \times 4 = 96{,}000$ inch-pounds.

5. What is the moment about the neutral axis of the forces of Problem 3?

Ans. $\left(36{,}000 \times 4 - 9{,}600 \times \frac{8}{3}\right) 2 = 236{,}800$ inch-pounds.

6. Find the total moment about the neutral axis of the forces of Problem 2. To find the moment of a trapezoidal wedge, subtract the moment of one triangular wedge from the moment of another.

Ans. $M = 25{,}600 + 14{,}400 = 40{,}000$ inch-pounds.

The total resisting moment of a rectangular section is

$$M = \frac{Sbd}{4} \times \frac{2d}{3} = \frac{Sbd^2}{6}. \qquad \text{Formula XII.}$$

By means of Formula XII, the maximum fiber stress in a beam of rectangular section may be calculated when the loads are known, or the load may be calculated for any given allowable stress.

Example

A 4-inch by 6-inch cantilever carries a load of 240 pounds on the free end. Find the unit stress in the top and bottom fibers at a section 5 feet from the free end.

Horizontal dimensions are given first. A 4-inch by 6-inch beam is 4 inches wide and 6 inches deep. Since unit stresses are required in pounds per square inch, the moment must be in inch-pounds.

$$M = \frac{Sbd^2}{6},$$

$$S = \frac{6M}{bd^2} = \frac{6 \times 14{,}400}{4 \times 36} = 600 \text{ pounds per square inch.}$$

Problems

7. A 6-inch by 10-inch beam is 15 feet long. It is supported at the ends and carries a load of 2,400 pounds at the middle. Find the unit stress in the outer fibers at the dangerous section caused by this load.

Ans. 1,080 pounds per square inch.

8. In Problem 7, what is the total tension in the lower half of the beam at the dangerous section?

Ans. 16,200 pounds.

9. An 8-inch by 12-inch beam is 20 feet long. It is supported at the ends and carries a load at the middle which makes the maximum unit stress at the middle 1,000 pounds per square inch. What is the moment at the middle and what is the load?

60. Fiber Stress in a Beam of Any Section.—The methods of Article 59 are not convenient to apply to sections other than rectangles. There is a general method, however, which applies to sections of any form. The moment of inertia and the center of gravity of plane figures are important factors in this method. As these are given in handbooks for many geometrical figures, a great saving of labor is gained by their use.

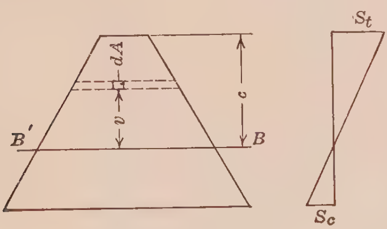


FIG. 86.—Beam section.

Figure 86 may be regarded as representing a section of any form. BB' is the neutral axis. An element of area dA is at a distance v from the neutral axis. (The letter v will be used to represent distance from the neutral axis in a section, and y will be reserved to represent deflection of the axis from its original position.) The area dA may be infinitesimal in two dimensions or it may extend entirely across the section parallel to the neutral axis as shown by the dotted lines.

Since the unit stress varies as v, it may be represented by kv, in which k is the unit stress at unit distance from the neutral

axis. The total stress on the element dA is the unit stress times the area.

$$\text{Total stress} = kvdA. \tag{1}$$

The moment of this stress on dA about the neutral axis,

$$dM = kv^2dA. \tag{2}$$

Since v^2 is positive when v is positive or negative, the sign of increment of moment is the same whether the element is above or below the neutral axis.

$$M = k\int v^2dA = kI, \tag{3}$$

in which I is the moment of inertia of the section with respect to the neutral axis. Since

$$s = kv,\ k = \frac{s}{v}, \tag{4}$$

which substituted in (3) gives

$$M = \frac{sI}{v}. \qquad \text{Formula XIII.}$$

Formula XIII gives the unit stress at any distance from the neutral axis. The most important stress is the stress in the extreme outer fibers where v is a maximum and the unit stress is the greatest. If this maximum unit stress be represented by S and the distance to the outer fiber from the neutral axis be represented by c, the formula becomes

$$M = \frac{SI}{c},$$

This formula is so important that it is desirable to memorize it also in the form

$$S = \frac{Mc}{I}. \qquad \text{Formula XIV.}$$

61. Location of the Neutral Axis.—The values of I and c in Formula XIII depend upon the location of the neutral axis. This is found from the condition that the total tensile stress across the part of the section on one side of the neutral axis is equal to the total compressive stress across the part of the section on the other side of the axis. On an element dA,

$$\text{total stress} = kvdA. \tag{1}$$

$$\text{Total stress on entire section} = k\int vdA = 0. \tag{2}$$

The constant k is not zero when the beam is bent, consequently $\int vdA$ must be zero.

The center of gravity of a plane area is given by

$$\bar{v} = \frac{\int vdA}{A}; \tag{3}$$

$$\bar{v}A = \int vdA = 0. \tag{4}$$

Since A is not zero

$$\bar{v} = 0. \tag{5}$$

The neutral axis of a beam of any section passes through the center of gravity of the section.

62. Section Modulus.—The expression $\frac{I}{c}$, in which c is the distance from the neutral axis to the extreme outer fiber, is called the *section modulus* or *modulus* of the *section*. If the section modulus is represented by Z, Formula XIV becomes

$$S = \text{unit stress in outer fibers} = \frac{M}{Z}. \tag{1}$$

The values of the section moduli of rolled shapes are given in the handbooks of the steel companies. These handbooks also give the moduli for the principal geometric sections. (See "Elements of Sections" in Carnegie, and "Properties of Various Sections" in Cambria. In Carnegie, the section modulus is represented by S, and stress in the outer fiber by f.)

For a rectangular section, $I = \frac{bd^3}{12}$ and $c = \frac{d}{2}$,

$$Z = \frac{I}{c} = \frac{bd^2}{6}. \tag{2}$$

When this value of Z is substituted in Formula XIII,

$$M = \frac{Sbd^2}{6},$$

which is Formula XII.

Problems

1. Look up in the handbook the section modulus of a 10-inch, 25-pound I-beam. Check by dividing the moment of inertia given in the table by the distance from the neutral axis to the extreme fiber.

2. A 12-inch, 35-pound I-beam with its web vertical is subjected to a load which makes the unit stress in the outer fibers 12,000 pounds per square inch. Find the bending moment in inch-pounds.

Ans. $M = 38 \times 12{,}000 = 456{,}000$ inch-pounds (Cambria).
$M = 37.8 \times 12{,}000 = 453{,}600$ inch-pounds (Carnegie).

3. What is the section modulus of a triangle of which the base is 6 inches and the altitude is 8 inches?

Ans. $Z = 16$ in.^{3}

4. A 15-inch, 50-pound I-beam is 20 feet long. It is supported at the ends and carries a load of 16,000 pounds 8 feet from the left end. Find the fiber stress in the outer fibers at the dangerous section caused by this load.

Ans. $S = 14{,}288$ pounds per square inch.

5. Solve Problem 4 with the weight of the beam included.
6. Find the section modulus of an I-beam for a span of 20 feet to carry 1,200 pounds per foot, including its own weight, and a load of 6,000 pounds 4 feet from the right support, if the allowable fiber stress is 16,000 pounds per square inch.

Ans. Maximum moment = 871,200 inch-pounds. $Z = 54.45$. The section modulus of a 15-inch, 42-pound I-beam is 58.9.

7. An I-beam, 24 feet long, is supported at the left end and 4 feet from the right end. It carries a load 600 pounds per foot including its own weight, a load of 4,800 pounds 5 feet from the left end and a load of 1,200 pounds 16 feet from the left end. Find the I-beam required if the allowable fiber stress is 16,000 pounds per square inch.

Ans. 12-inch, 31.5-pound I-beam (Cambria).
12-inch, 27.9-pound I-beam (Carnegie).

8. Solve Problem 7 if an additional load of 6,000 pounds is placed on the right end of the beam.
9. Find the total safe load, uniformly distributed, on an 8-inch by 10-inch white oak beam, which is 12 feet long and is supported at the ends. Use the allowable unit stresses recommended by the American Railway Engineering Society.
10. What should be the depth of a beam of long-leaf yellow pine which is 15 feet long and 8 inches wide, is supported at the ends, and carries a load of 2,400 pounds at the middle?
11. Solve Problem 4 of Article 59 by means of Equation (1).
12. Solve Problem 5 of Article 59 by means of Equation (1).
13. A 15-inch, 50-pound I-beam has a 14-inch by ½-inch plate riveted to the lower flange. Find the distance of the center of gravity of the combination from the center of the I-beam section, and find the moment of inertia and section modulus with respect to the horizontal axis through the center of gravity.
14. The beam of Problem 13 is 20 feet long, is supported at the ends, and carries a uniformly distributed load. Find the total load if the allowable unit stress is 15,000 pounds per square inch. Where is a beam of this kind used?
15. An 18-inch standard I-beam weighing 55 pounds per foot, has a pair of 5-inch by 4-inch by ½-inch angles riveted on opposite sides of the web.

The upper surface of one leg of each angle is 12 inches below the top of the I-beam. Find the section modulus of the combination. Find the total safe load, uniformly distributed, if the beam is 24 feet long and the allowable unit stress is 15,000 pounds per square inch.

16. Show that the section modulus of a circular section of radius a is $\frac{\pi a^3}{4}$.

17. A solid rectangular beam is b wide and d high. With the same unit stress, show that the ratio of the load with d vertical to the load with b vertical is the ratio of d to b.

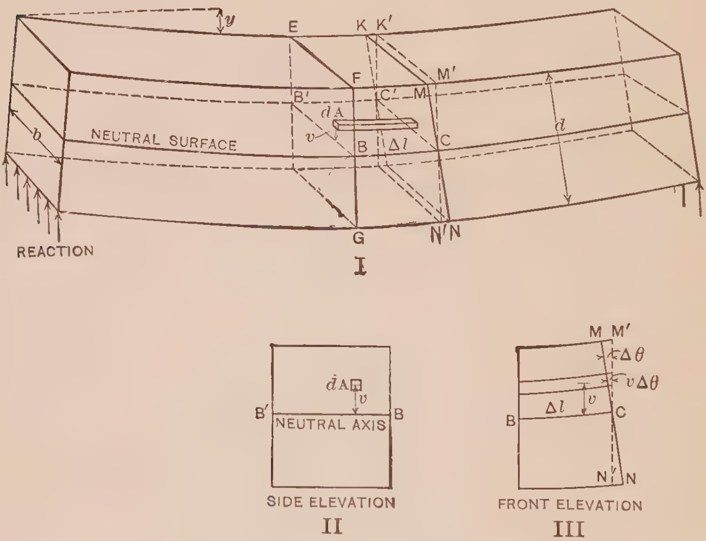


FIG. 87.—Deformation of a bent beam.

63. Relation of Stress to Deformation.—Figure 87 represents a bent beam with the concave side upward (the amount of bending is exaggerated). *EFG* is a plane section with neutral axis BB'. The broken lines $K'M'$, $M'N'$ indicate the position, before the beam was bent, of a plane section parallel to *EFG* at a distance Δl from that section. The section *EFG* may be regarded as fixed in position and direction, and the parts of the beam on both sides of *EFG* may be considered as bent upward. The plane $K'M'N'$ is rotated about its neutral axis CC' through an angle $\Delta\theta$ to the position *KMN*. (There is also a slight shift upward, but this does not affect the problem.) Since there is no elongation in the neutral surface, the distance between the neutral axes BB' and CC', measured along the curved surface, remains unchanged.

It is assumed that a plane section in a beam remains plane when the beam is bent, therefore the section KMN is a plane. A filament of cross-section dA extends from the plane EFG to the plane KMN, at a distance v from the neutral surface. When the beam is bent, and the plane KMN is turned through the angle $\Delta\theta$, this filament is shortened an amount $v\Delta\theta$. A similar filament below the neutral surface will be elongated $v\Delta\theta$. The unit deformation of the filament is given by

$$\delta = \frac{v\Delta\theta}{\Delta l} \tag{1}$$

Under the condition that the deformations are such that no stress exceeds the proportional elastic limit, the unit stress varies as the unit deformation, and since the unit deformation varies as v, the unit stress varies as the distance from the neutral axis, as was assumed in Article 58.

Since unit stress is equal to $E\delta$, the unit stress above the neutral surface is given by

$$s_c = E_c v \frac{\Delta\theta}{\Delta l}. \tag{2}$$

Below the neutral surface

$$s_t = E_t v \frac{\Delta\theta}{\Delta l}. \tag{3}$$

In most cases **it is assumed** (and is practically true) **that the modulus of elasticity is the same in both compression and tension.** With this assumption,

$$s = Ev\frac{\Delta\theta}{\Delta l}, \tag{4}$$

is the expression for the unit stress in the beam, at any element of area.

On an element of area,

$$\text{total stress on } dA = Ev\frac{\Delta\theta}{\Delta l}dA. \tag{5}$$

The moment of this stress with respect to the neutral axis BB' is the product of the total stress on dA by the moment arm v;

$$dM = Ev^2\frac{\Delta\theta}{\Delta l}dA = E\frac{\Delta\theta}{\Delta l}v^2dA. \tag{6}$$

The total moment of all the filaments which make up the beam is the integral of dM over the section EFG. Integrating over this area, $\frac{\Delta\theta}{\Delta l}$ remains constant and

$$M = E\frac{\Delta\theta}{\Delta l}\int^{c_2} v^2dA = E\frac{\Delta\theta}{\Delta l}I, \tag{7}$$

in which c_1, c_2 are the distances of the lower and upper surfaces of the beam from the neutral surface, I is the moment of inertia of the cross-section EFG or KMN with respect to its neutral axis, and $\Delta\theta$ is the change in slope of the normal to the beam, or the change in slope of the tangent to the beam, in the length Δl.

Example

A 6-inch by 6-inch wooden beam rests on two supports, which are 50 inches apart, and overhangs each support 30 inches. Equal loads are placed on each overhanging end at 20 inches outside the supports. It is found that two sections between the supports, which are 40 inches apart and were parallel to each other before the beam was loaded, make an angle of 1 degree with each other after the loads are applied. Find the bending moment between the supports and find the loads, if E is 1,500,000 pounds per square inch.

$$M = \frac{1{,}500{,}000 \times 108 \times \pi}{180 \times 40} = 22{,}500\pi = 70{,}686 \text{ inch-pounds.}$$

The deformation at the top fibers is $\frac{\pi}{60}$ and the unit deformation is $\frac{\pi}{2{,}400}$. The unit stress at the outer fibers is $\frac{1{,}500{,}000\pi}{2{,}400} = 1{,}963.5$ pounds per square inch. The average compression in the upper half of the section is one-half of this and the total compression is $\frac{1{,}963.5 \times 18}{2} = 17{,}671$ pounds. The moment arm of the total compression is 2 inches. The total moment of the tension and compression is $17{,}671 \times 4 = 70{,}684$ inch-pounds.

Problems

1. In the example above, find the stress in the outer fiber by means of the moment and the section modulus.
2. Through what angle may a length of 3 feet of a steel plate 0.08 inch in thickness be bent, if the unit stress may not exceed 40,000 pounds per square inch?
3. A steel hack-saw blade, 0.023 inch thick, was bent 45 degrees in a length of 4 inches by a constant external moment. Find the stress in the outer fibers. *Ans.* 67,700 pounds per square inch.

Some idea of the magnitude of the quantities involved may be obtained from Fig. 88. This represents a beam 6 inches wide, 8 inches deep, and about 7 feet long, supported at two points about 80 inches apart. An extensometer (not shown) is attached at two points, F and M, 40 inches apart and 1 inch below the top of the beam. A second extensometer is attached at G and N, 1 inch from the bottom. Two loads of 4,000 pounds each are applied 16 inches from the supports. If the beam is made of timber, the deflection at the middle is about 0.08 inch. (This

deflection is too small to show in the drawing unless the scale is exaggerated.) The upper extensometer shows a shortening of about 0.0180 inch in the original length of 40 inches, and the lower extensometer shows an equal elongation. If the tension and compression are exactly equal, the neutral surface is midway

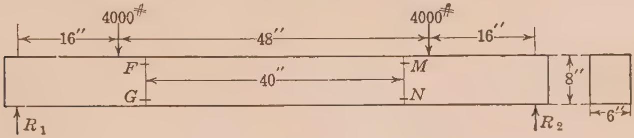


FIG. 88.—Arrangement for measuring linear deformation of a beam.

between the extensometers. If the readings are unequal, the location of the neutral surface may be found from the similar triangles, such as MCM', NCN' (Fig. 87), with MM' and NN' known from the extensometer readings, and the distance between the instruments equal to MN. In case the readings are each 0.0180 inch, showing that the neutral axis is at the middle of the section, 4 inches from the top, the compression in the top fibers is four-thirds as great as at M. The compression at 1 inch from the neutral surface is 0.0060 inch; and at a distance v it is $0.0060v$. The unit deformation at a distance v from the neutral axis is $0.00015v$.

It will be noticed that measurements are taken along the chord instead of along the arc, so that the readings are in error the amount of this difference. It may be shown, however, that the error is beyond the limits of the extensometer readings, and, therefore, makes no difference in the result.

An arrangement similar to Fig. 88 is largely used in the study of reinforced concrete beams. The beam is supported near the ends and two equal loads are applied at equal distances from the supports. If the weight of the beam is neglected, the moment is constant between the two loads. Measurements of elongation and compression, as described above, make it possible to locate the neutral axis. Frequently the loads are placed at the "third points," that is, at one-third the length from supports. Some experimenters place the loads near the ends and put the supports between the loads, as in the example above. This arrangement brings the tension fibers at the top, and makes it easier to study the fractures. In either case, the moment is constant for a considerable length of the beam. Measurements of elongation

and compression, combined with measurements of deflection on beams supported in this way, have amply verified the theory and proved the validity of the assumptions of this article.

Problems

4. A beam is tested as shown in Fig. 88. The points F and M are 40 inches apart and 6 inches above the similar points G and N. The compression reading on the upper instrument is 0.0198 inch, and the extension on the lower instrument is the same. What is the unit stress 4 inches above and 4 inches below the neutral surface, if E equals 1,500,000 pounds per square inch? *Ans.* 990 pounds per square inch.
5. In Problem 4, what is the unit stress at a distance unity and at a distance v from the neutral surface?
6. In an experiment similar to Problem 1 the upper extensometer shows a shortening of 0.0180 inch, and the lower extensometer an elongation of 0.0220 inch. The beam is 10 inches deep and the extensometers are 1 inch below the top and 1 inch above the bottom, respectively. How far is the neutral axis from the top of the beam? *Ans.* 4.6 inches.

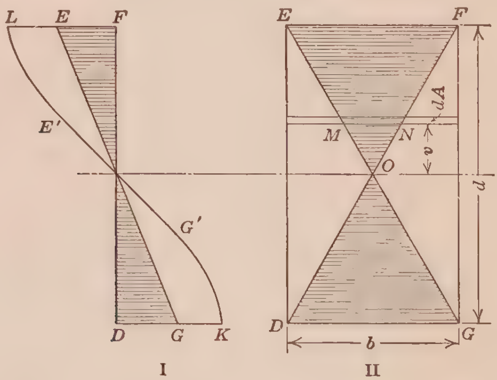


FIG. 89.—Stress distribution in a rectangular section.

64. Graphic Representation of Stress Distribution.—The unit stress in a beam, provided it does not exceed the proportional elastic limit, varies as the distance from the neutral axis. It may be represented by the straight line GE, Fig. 89, I. This straight line is really a part of the stress-strain diagram for the material in both tension and compression, with the vertical line DF as the X axis. If the unit stress is carried beyond the elastic limit, it may be represented by the line $KG'E'L$, which is also a stress-strain diagram with one scale changed.

In a beam of rectangular section, the total stress on any area dA, extending across the section, is proportional to the unit

stress. The shaded area of Fig. 89, I, may represent the total stress in a rectangular section as well as the unit stress in a section of any form. It is often convenient to represent total stress in a rectangular section by a figure similar to the shaded area in Fig. 89, II. This is really the same as Fig. 89, I, with oblique axes. The line EF represents the breadth of the section and also the total stress in the extreme outer fibers. It is evident, from the similar triangles, that the total stress on the area dA, extending across the section at a distance v from the neutral axis, will

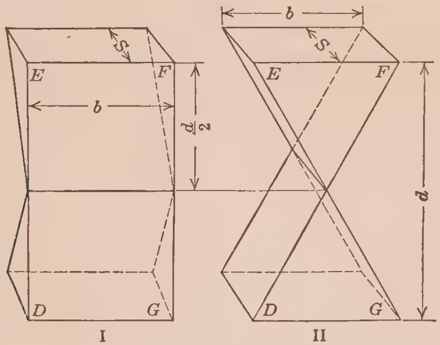


FIG. 90.—Stress distribution solids for a rectangular section.

be to the total stress at the top, as the length MN is to the length EF. The actual stress over the entire section is equal to a uniform stress of intensity equivalent to that in the outer fibers over the shaded area. If the cross-section is drawn to full scale, the area of the shaded triangle OEF gives the total stress above the neutral surface when the maximum stress is 1 pound per square inch. Likewise, the area OGD gives the total stress below the neutral surface. These triangular areas are equal in magnitude and opposite in sign, making the sum of the total stress zero.

The shaded triangles of Fig. 89, II, may be regarded as solids of uniform thickness. Figure 90, I, represents the distribution of stress in the same rectangular section by the method used in Article 59, but with both tension and compression on one side of the vertical plane which represents the section of the beam. It will be noticed that in Fig. 90, II, the width of the wedge varies as the distance from the neutral axis while the thickness, S, is constant; while in Fig. 90, I, the width is constant and equal to that of the section while the thickness varies as the distance from the neutral axis. From either figure the volume of one wedge is

$\frac{Sbd}{4}$ and its center of gravity, which is the location of the resultant force in that half of the section, is $\frac{d}{3}$ from the neutral axis. The moment of each wedge about the neutral axis is $\frac{Sbd^2}{12}$, and that of the two wedges representing tension and compression in a section is $\frac{Sbd^2}{6}$. Figure 90, I, shows the actual distribution of stress in a section. It may be called the *stress-distribution solid.* The shaded area of Fig. 89, II, represents a portion of the area of a section on which, if a uniform stress be applied equal to the unit stress in the outer fibers, the total stress and the moment with

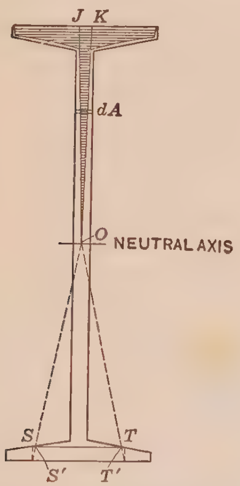


Fig. 91.—Stress distribution in an I-beam.

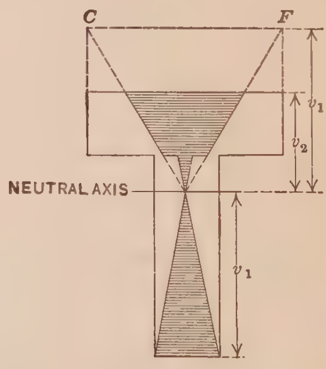


Fig. 92.—Distribution in a T-section.

respect to the neutral axis will be the same as that of the actual distribution. This is called the *stress-distribution diagram* or modulus figure.*

Figure 91 is the stress-distribution diagram for an I-beam section. For the rectangular part of the flange this diagram is drawn like Fig. 89. For a small area dA in the web, the length of dA is projected to the top of the section. The ends of the projection are the points J and K. Straight lines are drawn

* Goodman's "Mechanics Applied to Engineering," uses this name, and gives the figures for a great variety of sections.

from O to J and K respectively. The part of dA between these lines represents the total stress.

To get the stress on the triangular portion of the flange, consider the portion ST drawn (for convenience) in the lower flange. Project S and T on the lower line and connect the center O with the points thus found by means of the dotted lines. The portion $S'T'$ between these lines measures the total stress. A number of these lines will give the curved area required.

Figure 92 is the stress-distribution diagram for a T-section. The lower portion is constructed in the same way as Fig. 89. Above the top of the section is a line CF, which is as far from the neutral axis as the lowest fibers of the stem. The stress at the top of the flange is to the stress at the bottom of the stem as the distance v_2 is to the distance v_1. Lines drawn from a center on the neutral axis to the projections of the ends of the flange on CF enclose the stress-distribution diagram of the flange. The lines which enclose the diagram for the portion of the stem above the neutral axis are extensions of the lines below the neutral axis. The stress in this diagram is expressed in terms of the stress in the bottom fibers. It might be expressed in terms of the unit stress in the top fibers. For that diagram, lines would be drawn to the center from the ends of the upper edge of the flange. To draw the diagram for the stem a horizontal line would be drawn at a distance v_2 below the neutral axis. Below this horizontal line, the distribution diagram would extend beyond the sides of the stem.

Problems

1. Draw a stress-distribution diagram, similar to Fig. 92, for the T-section of Problem 2 of Article 59. Also draw a diagram which gives the stress in terms of the unit stress at the top of the flange. Show that the area of the upper portion is equal to the area of the lower portion. Multiply the area by the given unit stress to get the total stress. Find the moment of each area about the neutral axis, multiply by the unit stress, and compare the sum with the answer of Problem 6.
2. Construct the stress-distribution diagram for a 6-inch by 4-inch by 1-inch angle section for the neutral axis parallel to the 6-inch leg.

65. Stress beyond the Elastic Limit.—In the discussion of beams, it has been assumed that the unit stress is proportional to the unit deformation. This assumption is correct for allowable stresses, since these stresses are always considerably below the elastic limit. However, in order to know the factor of safety,

and to interpret correctly the results of tests in which beams are loaded to destruction, it is necessary to study the stresses and resisting moments in a beam when the stress in part of the fibers exceeds the elastic limit.

In Fig. 93, the horizontal lengths represent the unit stresses in a beam, while the vertical lengths represent unit deformations and distances from the neutral axis. The curved line from G to H is really a stress-strain diagram with the X axis vertical and the Y axis horizontal. Since the total stress in a rectangle varies as the unit stress, the shaded area of Fig. 93 may also

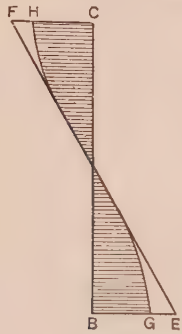


FIG. 93.—Stress distribution diagram beyond the elastic limit.

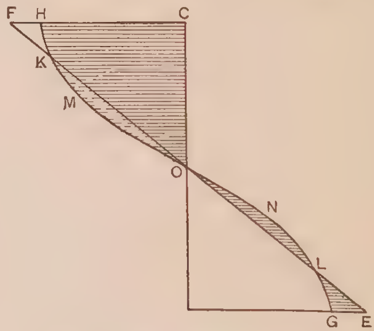


FIG. 94.—Actual and calculated unit stress.

represent the stress distribution, or *modulus figure*, for a rectangular section. The triangles of Fig. 93 may represent the stress distribution in a beam of rectangular section, in which the stress in the outer fibers is below the elastic limit. The shaded areas may represent the distribution in another beam, which has the same cross-section, the same deflection and unit deformation, the same modulus of elasticity for small stresses, but has a lower elastic limit. For about one-half the distance from the neutral axis to the outer fibers, the two diagrams coincide and the unit stress is the same for both beams. In the extreme outer fibers, the unit stress in the first beam is CF and in the second beam is CH. With the same deflection, the total stress and the total moment are greater in the first beam than in the second.

Figures 93 and 94 give comparisons of two beams which have equal sections and equal moduli of elasticity at small loads. In one beam, represented by the triangle, the elastic limit is above the unit stress in the outer fibers. In the other, represented by the shaded area, the stress in a considerable portion

of the beam is above the elastic limit. In Fig. 93, the *unit deformations* are the same for both beams, while the total stress and the moment represented by the triangle are greater than those of the shaded area. In Fig. 94, the unit deformations are different, while the resisting moment of the curved area $OMKHC$ is equal to that of the triangular area OFC. From the center of the section to the point K, the curve lies outside of the straight line. The unit stress in the fibers near the neutral surface is greater than if the modulus were constant. In the outer fibers, the unit stress is less than if the modulus were constant and the resisting moment were the same. The moment of the dotted area OMK (or the shaded area ONL) is equal to the moment of KFH or LGE.

Figure 94 might apply to two steel beams which have the same section, the same modulus of elasticity of 30,000,000 pounds per square inch, but different elastic limits and unit deformations. The unit deformation in one of these beams may be 0.003 in the outer fibers and the elastic limit be above 90,000 pounds per square inch. The length would then represent the unit stress in the outer fibers and the triangle OFC would be the distribution diagram for a rectangular section. The curve $OMKH$ would represent the unit stress in the other beam for which the elastic limit is about 30,000 pounds per square inch. The unit deformation in the outer fibers of this beam would be over 0.004. At about one-fourth the distance from the neutral axis to the outer fibers, the stress would be proportional to the unit deformation and would be one-third greater than that of the first beam. At the extreme outer fibers, the stress in this beam would be to the stress in the first beam as the length CH is to the length CF.

66. Modulus of Rupture.—When a beam is broken by bending, the stress-distribution diagram for a rectangular section, $OMKH$ (Fig. 94), is similar to the *complete* tension or compression curve of the material. The actual unit stress in the outer fibers is less than that obtained from the equation

$$S = \frac{Mc}{I}. \qquad \text{Formula XIV.}$$

in the ratio of $CH : CF$ (Fig. 94). The *calculated* value of the stress in the outer fibers computed from Formula XIV is called the *modulus* of *rupture*, or the transverse ultimate strength of the material. It is also called the extreme fiber stress in bending.

Another factor which makes the calculated modulus of rupture different from the actual unit stress is the shifting of the neutral axis. In sections which are not symmetrical with respect to the axis, the remote fibers on one side reach the elastic limit before those on the other. Figure 95 represents a T-section. Figure 95, II, shows part of the stress-strain diagram for both tension and compression. Figure 95, III, shows the distribution of stress for small deformation which produces no stress beyond the proportional elastic limit. The neutral axis passes through the center

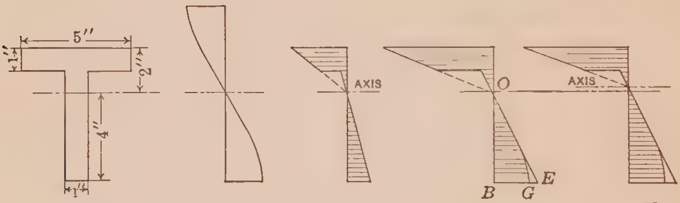


Fig. 95.—Displacement of the neutral axis.

of gravity of the section. Figure 95, IV, shows the distribution when the deformation is doubled, on the assumption that the neutral axis is not shifted. The lower half of the stem has passed the elastic limit and the unit stress in it is *not* proportional to the distance from the neutral axis. The shaded area below the axis is smaller than that above (which is equal to the triangle OBE) and consequently the neutral axis cannot pass through the point O, but must be moved upward away from the center of gravity of the section. Figure 95, V, is the actual diagram with the axis shifted. The area above the axis is diminished and that below increased. With a still greater deformation the upper fibers will also pass the elastic limit, and it may happen that, with some forms of stress-strain diagrams, the neutral axis may move backward toward the center of gravity of the section.

The neutral axis may be shifted in a *symmetrical section,* if the tension and compression curves are not alike. In cast iron, for instance, the stress-strain diagram for tension differs greatly from the diagram for compression. The compressive strength of cast iron is three or four times as great as the tensile strength. Beams of this material should be made of T-section or equivalent, and so loaded as to bring the stem in compression and the flange in tension. The remote fibers on the compression side should be

two or *three* times as far from the center of gravity of the section as those on the tension side.

While the modulus of rupture does not give the actual unit stress in the outer fibers, it makes it possible to compare stresses in similar *sections*. If the modulus of rupture of a given material is obtained from tests of beams of *rectangular* section, this modulus may be used in computing the ultimate strength of beams of this material of *any rectangular* section. The results may also be used with little error for beams of other shapes, provided they are symmetrical with respect to the neutral axis. With unsymmetrical sections, such as angles, it is better to make tests and obtain the modulus of rupture for each shape.

The student will remember, however, that these statements apply to the stress beyond the elastic limit. Since allowable stresses are below the elastic limit, Formula XIV is strictly correct for allowable loads. The change in the stress-distribution diagram when the stress passes the elastic limit *affects* the *factor* of *safety only.*

Strictly speaking, ductile materials, such as soft steel, have no modulus of rupture, since beams of such material may be bent double without breaking.

Problems

1. A spruce beam, 1¾ inches square, tested at the Bureau of Standards, was supported at points 24 inches apart and loaded midway between the supports. The beam broke at the middle under a load of 1,071 pounds. The deflection at the middle was 0.335 inch when the load was 1,012 pounds. Find the modulus of rupture.
Ans. 7,194 pounds per square inch.

2. A rectangular bar of cast iron, 1.04 inches wide and 0.80 inch thick, was placed on supports 12 inches apart and was broken by a load of 1,635 pounds midway between the supports. Find the modulus of rupture.

3. An ash beam, 2.51 inches square, which was tested at Watertown Arsenal,* was supported at points 24 inches apart, and failed under a load of 3,550 pounds at the middle. Find the modulus of rupture.

4. Another ash beam, 2.50 inches square, tested in the same way as that of Problem 3, failed under a load of 5,253 pounds. Find the modulus of rupture. *Ans.* 12,103 pounds per square inch.

5. A beam of short-leaf yellow pine, tested by Prof. A. N. Talbot at the University of Illinois, had the following dimensions: breadth, 7.12 inches; depth, 16.25 inches; distance between supports, 13 feet 6 inches. Two equal loads were applied at points 4 feet 6 inches from the supports, making the bending moment constant and the shear zero between these

* Tests of Metals, Etc. 1915, page 70.

points (if the weight of the beam is neglected). The beam broke by tension in the outer fibers between the loads when each load was 27,500 pounds. Find the modulus of rupture.

Ans. 4,739 pounds per square inch.

6. A beam of long-leaf yellow pine 7 inches wide and 14 inches deep, supported and loaded as in Problem 5, broke under a *total* load of 37,300 pounds. What was the ultimate bending strength of this timber?

Ans. 4,400 pounds per square inch.

7. A second beam of long-leaf yellow pine 7.0 inches by 12.1 inches, supported and loaded as above, broke under a total load of 52,900 pounds. What was the ultimate bending strength of this timber?

Ans. 8,362 pounds per square inch.

Problems 5, 6 and 7 are from *Bulletin* No. 41 of the University of Illinois Engineering Experiment Station.

Professor C. B. Upton has devised a method by which the actual unit stress in a beam of rectangular section may be computed from the curve which gives the load and deflection of the beam. (See Upton's "Materials of Construction," page 78.)

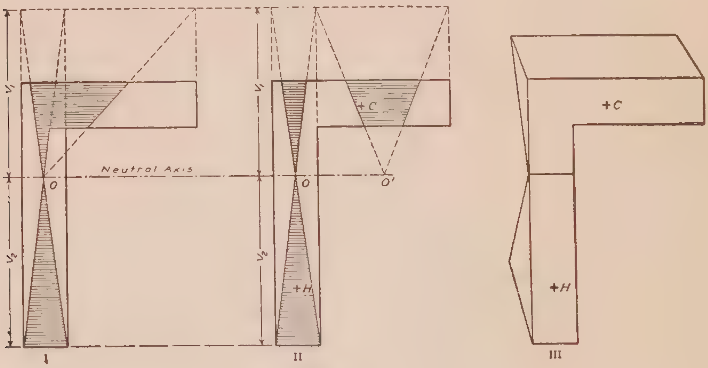


FIG. 96.

67. Neutral Axis for an Unsymmetrical Section.—If the sections of a beam are not symmetrical with respect to some vertical plane through their centers of gravity, the beam will not bend directly downward under a vertical load. Figure 96 shows two methods of drawing the stress-distribution diagram for an angle section, which is assumed to be bent in a vertical plane. Figure 96, I, is the usual method, with the center O at the middle of the vertical leg on the horizontal line through the center of gravity of the section. In Fig. 96, II, there are two centers for the upper portion. One center O lies in the vertical line through the center of the vertical leg and the other center O' lies in the

vertical line through the center of gravity of the portion of the horizontal leg which lies to the right of the vertical leg. Each method gives the total stress in terms of the unit stress in the fibers at the bottom of the beam, and each shows the true distance of the resultant forces from the horizontal axis. The true position C of the resultant force in the upper portion is at the center of gravity of the triangular and the trapezoidal areas of Fig. 96, II. It is at the center of gravity of the upper wedge of the stress distribution solid of Fig. 96, III.

The forces H and C perpendicular to the plane of the paper form a couple, which is the resisting moment. The plane of this couple is not vertical. To bend the beam in a vertical plane the external moment must lie in the same plane as the resisting moment. A vertical load will not bend a beam of this form in a vertical direction. If there were two equal angles fastened together with their vertical legs back to back, then the combination would be symmetrical with respect to a vertical axis, and the neutral axis for vertical loads would be horizontal.

In Fig. 97, $ABCD$ is a rectangular section with diagonal horizontal. It may be regarded as the end of a cantilever perpendicular to the plane of the paper. If a vertical load P is placed on the end of this cantilever, the deflection will not be vertically downward, but the section will be displaced into a position such as shown in the figure.

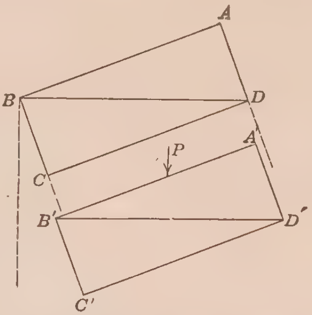


FIG. 97.—Rectangular beam with load perpendicular to diagonal.

68. Bending Moment Not in Plane of Principal Moment of Inertia.—Figures 96 and 97 are special cases of the general problem in which the bending moment does not lie in the plane of one of the two principal moments of inertia. If an axis perpendicular to the plane of the bending moment is an axis for which the moment of inertia of the section is a maximum or a minimum, then the beam will bend around this axis and the deflection will be in the direction of the applied forces. For instance, when an I-beam is placed with the web vertical or a rectangular beam is placed with the long sides of the rectangle vertical, the axis of maximum moment of inertia is horizontal, and a vertical load

will deflect the beam vertically downward. When the section of a beam is a square, an equilateral triangle, or any other plane figure with all sides equal and all angles equal, the moment of inertia is the same for all axes through the center of gravity, and the neutral axis is always perpendicular to the plane of the applied forces.

The method of finding the unit stress when the bending moment is not in the plane of a principal axis of inertia is very simple. *Resolve the bending moments or the applied forces into two components perpendicular to the two principal axes of inertia, and compute the stress separately for each component. The actual stress at any point is the algebraic sum of the stresses caused by the two components.*

Example

A cantilever beam of rectangular section is 4 inches by 3 inches and is 5 feet long. The beam is placed with the 4-inch faces at 30 degrees with the horizontal and a load of 120 pounds is put on the free end. Find the unit stress at each corner, and find the direction of the neutral axis.

The load of 120 pounds is resolved into 103.9 pounds perpendicular to the 4-inch faces and 60 pounds perpendicular to the 3-inch faces. From the first component, the stress in AB and in CD of Fig. 98 is

$$S = \frac{103.9 \times 60}{6} = 1{,}039 \text{ pounds per square inch.}$$

From the second component,

$$S = \frac{60 \times 60}{8} = 450 \text{ pounds per square inch.}$$

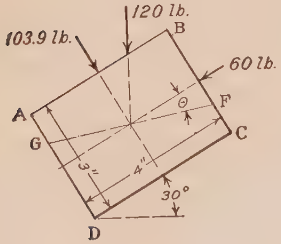


Fig. 98.

At B both stresses are tensile and at D both are compressive. The unit stress at these corners is 1,039 + 450 = 1,489 pounds per square inch. At A the stress caused by the 60-pound component is compressive, while that caused by the other component is tensile. The tensile stress at A and the compressive stress at C are 1,039 − 450 = 589 pounds per square inch.

The location on the line CB of the point F at which the stress is zero is found by dividing the distance from C to B in the ratio of 589 to 1,489.

$$CF = \frac{589 \times 3}{589 + 1{,}489} = \frac{1{,}767}{2{,}078} = 0.850 \text{ inch.}$$

At G on the line AD at a distance of 0.850 inch from A, the unit stress is zero. The line GF through the center of the section is the neutral axis.

The angle θ between the neutral axis GF and a line parallel to the 4-inch faces is given by

$$\tan \theta = \frac{0.65}{2} = 0.325;\ \theta = 18 \text{ degrees, nearly.}$$

The neutral axis makes an angle of 12 degrees with the horizontal.

Problems

1. A rectangular cantilever, 6 inches by 10 inches, is 6 feet long. The 10-inch faces make an angle of 20 degrees with the vertical. The beam carries a load of 1,200 pounds on the free end. Find the unit stress at each corner and the angle which the true neutral axis makes with the horizontal. *Ans.* 1,304 pounds per square inch; 320 pounds per square inch; 25 degrees, 16 minutes.

2. A rectangle of sides b and d is placed with one diagonal horizontal andsubjected to a vertical load. Find the fiber stress at the corners A and C, Fig. 99.

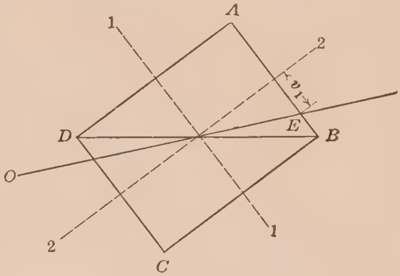


FIG. 99.—Rectangular beam.

$$Ans.\ S = \frac{6M\sqrt{d^2 + b^2}}{b^2d^2}.$$

3. Show that the result of Problem 2 is the same as would be found if the neutral axis coincided with the horizontal diagonal DB.

4. In Problem 2 find the unit stress at the corners B and D, and the direction of the neutral axis.

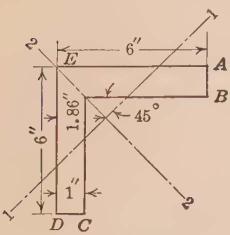


FIG. 100.—Angle section.

5. A 6-inch by 6-inch by 1-inch standard angle, 10 feet long, is used as a beam supported at the ends. The angle is placed with legs horizontal and vertical and a load of 1,000 pounds is applied at the middle, over the center of gravity of the section. Find the unit stress at the corners. Here the principal axes are 1–1 for which the moment inertia is 14.78, and 2–2 for which the moment of inertia is 56.14. The bending moment for each axis is $15{,}000\sqrt{2}$.

$$\text{Unit stress at } E = \frac{15{,}000 \times \sqrt{2} \times 1.86 \times \sqrt{2}}{14.78} = 3{,}775 \text{ lb./in.}^2$$

Ans. Unit tensile stress at $C = 3{,}329 + 1{,}336 = 4{,}665$ pounds per square inch.

69. Bending Moments in Different Planes.—It frequently happens that a beam is subjected to forces which are not all parallel. If the beam has a circular or square section so that the moment of inertia is the same in all directions, the resultant mo-

ment may be calculated at any section and this moment may be used to find the fiber stress. If the two principal moments of inertia are not equal, the forces or moments should be resolved in the directions of the principal axes, and the stress at any point calculated as in Article 68.

Example

A horizontal cantilever 5 feet long carries a load of 120 pounds per foot and is subjected to a horizontal pull, perpendicular to its length, of 400 pounds at the free end. Find the expressions, in inch-pounds, for the moment at any section.

Ans. $M_y = 5x^2$; $M_z = 400x$; Resultant $M = 5x\sqrt{x^2 + 6{,}400}$, in which M_y is the moment in the vertical plane and M_z is the moment in the horizontal plane.

Problems

1. In the example above, find the direction and magnitude of the resultant moment at the fixed end.
Ans. 30,000 inch-pounds in a plane at an angle of 36 degrees 52 minutes with the horizontal.

2. If the cantilever of Problem 1 is of circular section, 4 inches in diameter, find the maximum fiber stress. *Ans.* 4,775 pounds per square inch.

3. In Problem 1 find the magnitude and direction of the resultant moment at 30 inches from the free end.
Ans. 12,816 inch-pounds at angle of 20 degrees 34 minutes with the horizontal.

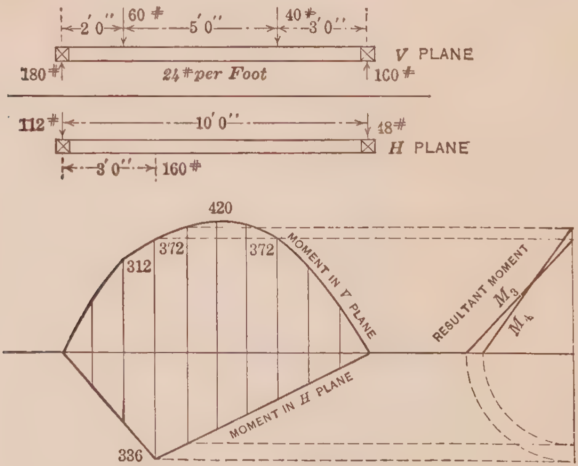


FIG. 101.—Beam with horizontal and vertical loading.

4. A horizontal shaft 10 feet long, weighing 24 pounds per foot, is supported at the ends and carries a vertical load of 60 pounds 2 feet from the left end and a vertical load of 40 pounds 3 feet from the right end. A hori-

zontal force of 160 pounds, perpendicular to the shaft, is applied 3 feet from the left end. Find the resultant moment at 3 feet from the left end and at the middle.

Ans. 501 foot-pounds at 3 feet; 484 foot-pounds at 5 feet.

5. Write an expression for the moment in each plane and for the resultant moment between 3 feet and 7 feet from the left support. Differentiate the expression for the resultant moment to derive an equation for finding the position of maximum moment.

Figure 101 shows the diagrams for the moment in the vertical and horizontal planes for Problem 4. The maximum moment in the vertical plane is at 5 feet, and the maximum in the horizontal plane is at 3 feet. The maximum resultant moment is between 3 feet and 5 feet. The resultant moment at any section may be determined graphically from the diagonal of the right-angled triangle, the legs of which are the horizontal and vertical moments.

Problem

6. A horizontal shaft 12 feet long, supported at the ends, carries a vertical load of 600 pounds 3 feet from the left end, and a horizontal pull of 400 pounds perpendicular to its length at 6 feet from the left end. Find the location of the maximum resultant moment.

TABLE X.—ALLOWABLE UNIT BENDING STRESSES

Material	Unit stress in pounds per square inch
Structural steel	16,000
Wrought iron	12,000
Cast steel	16,000
Cast iron in tension	3,000
Cast iron in compression	15,000
Concrete in compression:	
1:2:4	650
1:3:6	450
Long-leaf yellow pine	1,200
Douglass fir	1,100
White oak	1,100
Norway pine	900

Miscellaneous Problems

1. A 4-inch by 6-inch beam, 18 feet long, is supported at the left end and 3 feet from the right end, and carries a load of 60 pounds per foot and a load of 360 pounds on the right end. Find the maximum moment, the maximum fiber stress, and the location of the point of zero moment.
2. Solve Problem 1 if the load at the right end is 60 pounds instead of 360 pounds.

3. In Problem 1, what is the unit stress at 5 feet from the left end at 1 inch from the top?
4. A 15-inch, 42-pound I-beam is 20 feet long and is supported at the ends. It carries a load at the middle which brings the unit stress at the middle section at a point 2 inches from the top to 11,000 pounds per square inch. Find the load.
5. The safe load at the middle of a 6-inch by 8-inch beam, which is 15 feet long and is supported at the ends, is 1,200 pounds. What is the safe load on a 6-inch by 6-inch cantilever of the same material which is 5 feet long and carries the load on the end?
6. How does the strength of a 4-inch by 6-inch beam compare with the strength of a 6-inch by 8-inch beam of the same material and the same length? Solve without writing.
7. What should be the size of a square beam of long-leaf yellow pine, which is 12 feet long, is supported at the ends, and carries a load of 1,800 pounds at the middle?
8. An 18-inch, 55-pound I-beam is 20 feet long and is supported at the ends. It carries a load 6 feet from the left end and an equal load 6 feet from the right end. When these loads are applied, a length of 40 inches at the lower surface is stretched 0.0244 inch. Find the loads.
9. A 12-inch, 31.5-pound I-beam rests on two supports which are 10 feet apart and overhangs the right support 3 feet and the left support 4 feet. A load of 12,000 pounds is placed on the right end and a load of 9,000 pounds is placed on the left end. If the beam is horizontal midway between the supports, what angle does it make with the horizontal at each support? Neglect the weight of the beam.
10. A 10-inch, 25-pound I-beam, 10 feet long, is supported at the ends and carries a load of 6,000 pounds including its own weight, uniformly distributed. A horizontal push of 240 pounds, perpendicular to the length of the beam is applied 4 feet from one end. Find the unit stress at the extremities of the flanges at the middle.
11. Two 10-inch, 15-pound channels are placed parallel in a horizontal position. They are supported on a wall at one end and held up by an eye-bar which is placed between them at the other end and connected by a pin. The channels are 10 feet long from the wall to the center of the pin. The eye-bar makes an angle of 45 degrees with the horizontal. A load is placed on the channels 6 feet from the wall. The allowable bending stress is 12,000 pounds per square inch, the allowable shearing stress is 10,000 pounds per square inch, and the allowable bearing stress is 16,000 pounds per square inch. Find the size of the eye-bar and the pin. Would it be necessary to rivet plates to the webs of the channels at the pin? If so, how thick should each plate be?
12. A square section with diagonal vertical has its section modulus increased by chamfering the top and bottom corners. What must be the dimensions of the triangular sections cut away, in terms of the sides, in order that section modulus may be a maximum? *Ans.* One-ninth the side.

CHAPTER VIII

DEFLECTION IN BEAMS

70. Relation of Moment to Curvature.—In Article 63, the relation of moment to change of curvature was found to be

$$M = EI\frac{\Delta\theta}{\Delta l} = EI\frac{d\theta}{dl}, \tag{1}$$

for infinitesimal lengths measured along the neutral surface of the bent beam. The angle $d\theta$ is the change in slope of the tangent to the neutral surface in the length dl. In Figs. 102

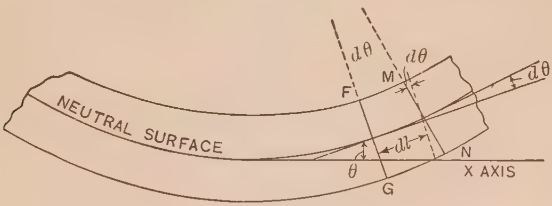


Fig. 102.—Curvature of beam.

and 87, two lines FG and MN are drawn perpendicular to the neutral surface at a distance dl apart. The broken line which intersects MN at the neutral surface is parallel to FG. The lines FG and MN make an angle $d\theta$ with each other, since they are normal to the neutral surface, and intersect at some point beyond the drawing, at a distance ρ from the neutral surface. This distance ρ is the radius of curvature of the neutral surface.

By geometry:

$$\rho d\theta = dl, \tag{2}$$

$$\frac{d\theta}{dl} = \frac{1}{\rho}. \tag{3}$$

Substituting in (1),

$$\frac{M}{EI} = \frac{1}{\rho}, \quad M = \frac{EI}{\rho}. \tag{4}$$

If M is constant, or if I varies as M, ρ is constant, and the curve of the beam is an arc of a circle which may be computed by trigonometry.

Problems

1. A 3-inch by 1-inch steel beam, 10 feet long, rests on two supports, each 30 inches from an end, and carries a load of 200 pounds on each end. (Fig. 103.) If the weight of the beam is neglected, what is the bending moment for the portion between the supports? If the modulus of elasticity is 30,000,000 pounds per square inch, what is the radius of curvature?

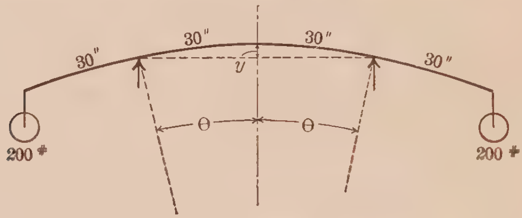


FIG. 103.—Curvature constant.

How much is the middle of the beam deflected upward above the supports? Solve for the deflection by geometry, assuming that the chord is equal to the arc. Solve also by trigonometry. If tables do not give the cosine with sufficient accuracy, calculate it by the series.

Ans. $M = 6{,}000$ inch-pounds; $\rho = 1{,}250$ inches; $y = 0.36$ inch.

2. A steel plate, 1 inch wide and 0.1 inch thick, is bent through an angle of 45 degrees in a length of 30 inches by a constant moment. If E is 30,000,000, what is the moment, the radius of curvature, and the deflection at the middle of the 30 inches?

Ans. $M = 65.45$ inch-pounds; $\rho = 38.2$ inches; $y = 2.90$ inches.

71. Change of Slope in Rectangular Coördinates.—When Equation(1) of Article 70 is integrated with θ and l as the variables, the integral gives the change of slope in radians. For most purposes, it is desirable to express the slope in rectangular coördinates.

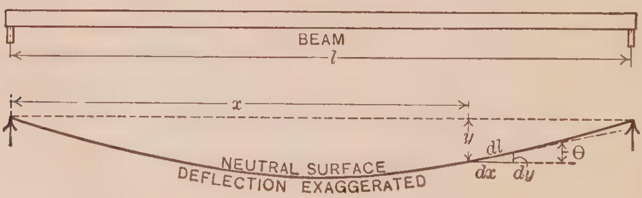


FIG. 104.—Slope and deflection.

The curved line of Fig. 104 represents the neutral surface of a bent beam with the deflection greatly exaggerated. In a floor beam, for instance, the deflection should not be more than 1 inch in 30 feet. A deflection as great as would be permissible in an engineering structure would not be noticeable in a drawing. For this reason, in the discussion of deflection which

follows, a beam has been drawn in each figure as it would appear to the eye, and then a heavy line has been drawn below to represent the position and slope of the neutral surface with all vertical distances magnified.

In Fig. 104, dl is a small length measured along the neutral surface, and dx is the projection of this length on the horizontal. $dx = dl \cos \theta$, in which θ is the slope of the tangent to the beam. With the small deflections allowable in an engineering structure, $\cos \theta$ is nearly unity and dx is practically equal to dl. For instance, if the slope were one part in 100 (which is relatively large) a triangle could be drawn with a base of 1 unit and an altitude of 0.01 unit. The hypotenuse of this triangle is $\sqrt{1.0001}$, which is 1.00005, and the error in the assumption that dx equals dl is one part in 20,000. When dx is substituted for dl, Equation (1) of Article 70 becomes

$$M = EI\frac{d\theta}{dx}. \tag{1}$$

Since θ is a small angle, its value in radians is practically equal to its tangent;

$$\theta = \tan \theta = \frac{dy}{dx}; \tag{2}$$

$$\frac{d\theta}{dx} = \frac{d^2y}{dx^2}; \tag{3}$$

$$M = EI\frac{d^2y}{dx^2}. \qquad \text{Formula XV.}$$

This formula may be derived in a slightly different manner, which will show the magnitude of the approximations. From the calculus, the reciprocal of the radius of curvature is

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left(1 + \left(\dfrac{dy}{dx}\right)^2\right)^{3/2}}. \tag{4}$$

When this expression is substituted in Equation (4) of Article 70, the result is

$$M = \frac{EI\dfrac{d^2y}{dx^2}}{\left(1 + \left(\dfrac{dy}{dx}\right)^2\right)^{3/2}}. \tag{5}$$

When $\dfrac{dy}{dx}$ is small, $\left(\dfrac{dy}{dx}\right)^2$ is much smaller and may be neglected.

The denominator of the second member of Equation (5) then becomes unity, and Equation (5) is equivalent to Formula XV. Mathematically, Equation (5) is reduced to Formula XV by multiplying by $\sec^3\theta$ or dividing by the $\cos^3\theta$, when $\tan\theta = \frac{dy}{dx}$.

In some problems in which the deflections are large, the exact formula of Equation (5) is required. The application of this equation will be considered in Appendix D. For all ordinary problems of beam and column deflection, Formula XV is ample. In the application of the formula the X axis is taken parallel to the unbent beam, and the deflection is so small that $\left(\frac{dy}{dx}\right)^2$ is negligible compared with unity.

72. Solution of the Differential Equation of Deflection.— Before solving Formula XV for the deflection of a beam or column, all the factors must be expressed in terms of x, y, and *constants.* The modulus of elasticity is constant, provided the unit stress does not exceed the proportional elastic limit. The formulas for deflection are valid only below this limit. For beams of uniform section, I is constant; for beams of variable section, it is expressed as a function of x. The moment is expressed as a function of x and y. In beams it is usually a function of x only, as in equations of Article 54.

When the expressions for M and I do not depend upon the deflection y, Formula XV becomes

$$\frac{d^2y}{dx^2} = \text{function of } x. \tag{1}$$

The first integration of Equation (1) gives $\frac{dy}{dx}$ as a function of x with the addition of an integration constant. If $\frac{dy}{dx}$ is known for any one value of x, these values may be substituted in the integral and the integration constant determined.

The second integration gives the deflection y as a function of x with the addition of a second integration constant. If y is known for some one value of x, these values may be substituted in the second integral and the second integration constant determined. The final integral with the integration constants given in terms of the loads and dimensions of the beam is called the *equation of the elastic line.*

If $\frac{dy}{dx}$ is not known for any one value of x, the first integration constant must be carried through the second integration, and the value of y must be known for two values of x to complete the solution.

73. Cantilever Loaded at Free End, by Double Integration.—Figure 105 represents a cantilever beam fixed at the right end and loaded at the left end. The beam was horizontal before the load was applied and remains horizontal at the wall when loaded.

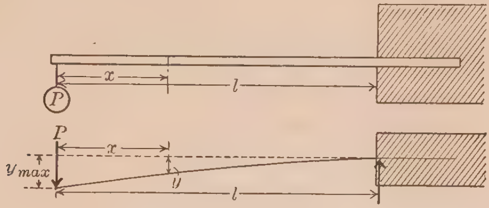


FIG. 105.—Cantilever with load on free end.

The origin of coördinates is taken at the position of the left end before the load was applied. The moment at *any* distance x from the origin is $-Px$. The differential equation is

$$EI\frac{d^2y}{dx^2} = -Px. \tag{1}$$

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + C_1. \tag{2}$$

At the wall, where $x = l$, the beam is horizontal and $\frac{dy}{dx} = 0$;

$$C_1 = \frac{Pl^2}{2}. \tag{3}$$

$$EI\frac{dy}{dx} = -\frac{Px^2}{2} + \frac{Pl^2}{2}. \tag{4}$$

$$EIy = -\frac{Px^3}{6} + \frac{Pl^2x}{2} + C_2. \tag{5}$$

At the wall $x = l$, $y = 0$;

$$0 = -\frac{Pl^3}{6} + \frac{Pl^3}{2} + C_2;$$

$$C_2 = -\frac{Pl^3}{3}. \tag{6}$$

$$EIy = -\frac{Px^3}{6} + \frac{Pl^2x}{2} - \frac{Pl^3}{3}; \tag{7}$$

$$y = -\frac{P}{6EI}(2l^3 - 3l^2x + x^3). \tag{8}$$

The maximum deflection is at the free end, where $x = 0$;

$$y_{\max} = -\frac{Pl^3}{3EI}. \qquad \text{Formula XVI.}$$

If $x = kl$, in which k is a fraction less than unity,

$$y = -\frac{Pl^3}{6EI}(2 - 3k + k^3). \qquad (9)$$

Problems

1. Find the slope of the beam at the free end, and at one-fourth the length from the free end.

   Ans. $\frac{dy}{dx} = \frac{Pl^2}{2EI}$, and $\frac{dy}{dx} = \frac{15Pl^2}{32EI}$.

2. Find the deflection at one-fourth the length and at one-third the length from the free end.

   Ans. $y = -\frac{27Pl^3}{128EI}$, and $y = -\frac{14Pl^3}{81EI}$.

3. A 4-inch by 6-inch wooden cantilever, 10 feet long, is deflected 1.2 inches at the end by a load of 240 pounds on the end. Find E and the maximum fiber stress.

   Ans. $E = 1{,}600{,}000$ pounds per square inch; $S = 1{,}200$ pounds per square inch.

4. In Problem 3, find the deflection 30 inches from the free end.

   Ans. $y = 0.759$ inch.

5. A cantilever with a load on the end is deflected 0.4 inch at the middle. Find the deflection at the end.

6. A 7-inch, 15-pound I-beam as a cantilever 5 feet long is deflected 0.136 inch at the end by a load of 2,000 pounds on the end. Find E and the maximum unit stress.

7. If E is 1,600,000 and the maximum allowable unit stress is 1,000 pounds per square inch, what is the maximum deflection on a beam 6 feet 8 inches long and 4 inches deep if the load is on the end and the sections are symmetrical with respect to a horizontal line through their centers of gravity?

8. A 15-inch I-beam as a cantilever 10 feet long is deflected 0.256 inch at the end by a load on the end. If E is 29,000,000 pounds per square inch, what is the maximum fiber stress?

   Ans. 11,600 pounds per square inch.

9. A cantilever of length l is fixed at the left end and carries a load P at the right end. With the origin of coördinates at the fixed end and with x positive toward the right, find the equation of the elastic line. (See Fig. 81.)

   Ans. $M = -P(l - x)$;

   $$EI\frac{dy}{dx} = \frac{P(l-x)^2}{2} - \frac{Pl^2}{2};$$

   $$EIy = -\frac{P(l-x)^3}{6} - \frac{Pl^2x}{2} + \frac{Pl^3}{6}.$$

10. Find the slope at the free end and at one-fourth the length from the free end by means of the results of Problem 9.

11. Find the deflection at the free end, at one-fourth the length from the free end, and at one-third the length from the free end by means of the results of Problem 9. Compare with Problem 2.

12. In Problem 9, write the moment $M = -Pl + Px$ and integrate for the equation of the elastic line. How do the integration constants differ from those of Problem 9? Solve Problems 10 and 11 by means of these integrals.

74. Cantilever with Load at Any Point.—Figure 106 shows a cantilever, which is fixed at the right end and carries a concentrated load P at a distance a from the left end. The portion to the left of the load remains straight (if the weight of the beam

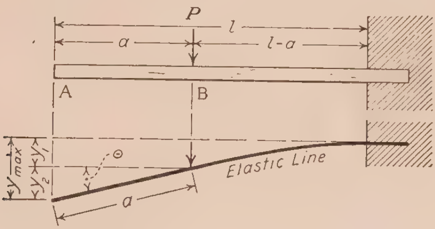


FIG. 106.

is neglected). The portion to the right of the load is a cantilever of length $l - a$ which is loaded at the end. The deflection of this portion is

$$y_1 = -\frac{P(l-a)^3}{3EI} \tag{1}$$

The additional deflection of the portion to the left of the load is $y_2 = -a \sin \theta$, in which θ is the slope at the load. For a small angle, $\sin \theta = \tan \theta = \frac{dy}{dx}$. From Equation (4) of Article 73, the slope under the load is

$$\frac{dy}{dx} = \frac{P(l-a)^2}{2EI}. \tag{2}$$

$$y_2 = -\frac{Pa(l-a)^2}{2EI};$$

$$y_{\max} = -\frac{P(l-a)^3}{3EI} - \frac{Pa(l-a)^2}{2EI} = -\frac{P}{6EI}(2l^3 - 3l^2a + a^3). \tag{3}$$

Problems

1. A cantilever of length l has a load P at one-fourth the length from the free end. Find the deflection at the end.

$$Ans. \quad -\frac{27Pl^3}{128EI}.$$

2. A cantilever of length l has a load P at one-third the length from the free end. Find the deflection at the free end.
3. A 4-inch by 6-inch cantilever, 6 feet long, carries 200 pounds 1 foot from the free end and 120 pounds 2 feet from the free end. Find the deflection at the free end and the deflection under the load of 200 pounds if E is 1,500,000. Find the maximum fiber stress.

75. Application of Maxwell's Theorem.—The deflection caused by several loads is equal to the sum of the deflections which would be produced by the loads acting separately. This statement, which is called Maxwell's theorem, may be regarded as an axiom which has been amply substantiated by experiment and by the theoretical investigations of special cases. From this theorem, an important proposition has been derived. This is: *If A and B are two points on a beam, the deflection at A which is caused by a given load at B is equal to the deflection at B which is caused by the same load at A.* (The proof of this proposition is given later in Chapter XVI.)

If x is substituted for a in Equation (3) of Article 74, the equation becomes identical with Equation (8) of Article 73. This identity verifies the proposition above for a cantilever with a concentrated load.

Problem

1. Compare the deflection of Problem 2 of Article 73 with that of Problem 1 of Article 74.

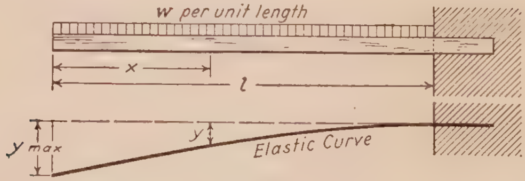


FIG. 107.—Cantilever with distributed load, fixed at right end.

76. Cantilever with Uniformly Distributed Load, by Double Integration.—Figure 107 shows a cantilever which is fixed at the right end and carries a load of w per unit length. The origin of coördinates is taken at the position of the free end before the

beam was bent. The moment at any section at a distance x from the free end is $-\dfrac{wx^2}{2}$

$$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}, \tag{1}$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1. \tag{2}$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wl^3}{6}. \tag{3}$$

$$EIy = -\frac{wx^4}{24} + \frac{wl^3x}{6} + C_2, \tag{4}$$

$$EIy = -\frac{wx^4}{24} + \frac{wl^3x}{6} - \frac{wl^4}{8}. \tag{5}$$

$$y = -\frac{w}{24EI}(3l^4 - 4l^3x + x^4), \tag{6}$$

which is the equation of the elastic line. At the free end,

$$y_{\max} = -\frac{wl^4}{8EI} = -\frac{Wl^3}{8EI}, \qquad \text{Formula XVII.}$$

in which $W = wl$ is the total distributed load.

Problems

1. Find the slope at the free end and at the middle.

Ans. $\dfrac{wl^3}{6EI}$; $\dfrac{7wl^3}{48EI}$.

2. Find the deflection at one-fourth the length and at one-half the length from the free end.

Ans. $y = -\dfrac{171wl^4}{2048EI}$; $y = -\dfrac{17wl^4}{384EI}$.

3. A 6-inch by 8-inch wooden cantilever, 10 feet long, is deflected $1\frac{5}{32}$ inch at the end by a load of 80 pounds per foot. Find E and the maximum unit stress.
4. What would be the deflection and stress for the beam of Problem 3, if the 6-inch faces were vertical? Solve without writing.
5. A given cantilever is deflected 1 inch at the end by a load of 1,200 pounds uniformly distributed. How much would it be deflected if the same load were placed on the free end? Solve without writing.
6. A cantilever, uniformly loaded, is deflected 1 inch at the end when the maximum unit stress is 1,000 pounds per square inch. What would be the deflection if a load were placed on the end which would make the same maximum unit stress? Solve without writing.

7. A cantilever with uniformly distributed load is fixed at the left end, as in Fig. 108. Derive the equation of the elastic line with the origin of coördinates at the left end. Show that

$$M = -\frac{w(l-x)^2}{2}; \tag{7}$$

$$EI\frac{dy}{dx} = \frac{w(l-x)^3}{6} - \frac{wl^3}{6}; \tag{8}$$

$$EIy = -\frac{w(l-x)^4}{24} - \frac{wl^3x}{6} + \frac{wl^4}{24}. \tag{9}$$

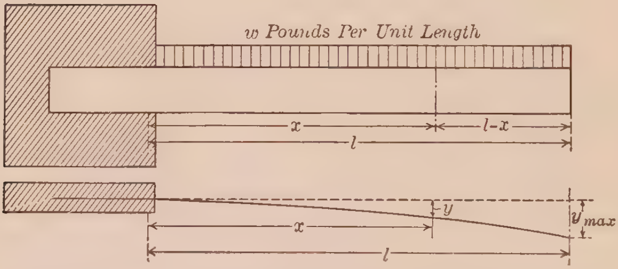


FIG. 108.—Cantilever with uniformly distributed load.

8. Find the deflection at the free end, at one-fourth the length from the free end, and at one-half the length from the free end by means of Equation (9) and compare with Problem 2.
9. Solve Problem 1 by means of the results of Problem 7.
10. Expand the expression for the moment in Problem 7 and integrate for the slope and the equation of the elastic line.
11. If $x = kl$, in which k is a fraction less than unity, show that the equation of the elastic line is

$$y = -\frac{Wl^3}{24EI}(3 - 4k + k^4).$$

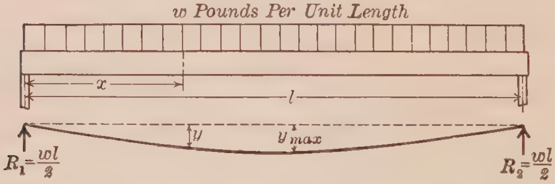


FIG. 109.—Supports at ends, load uniformly distributed.

77. Beam Supported at the Ends and Uniformly Loaded, by Double Integration.—In a beam supported at the ends and uniformly loaded, the end reactions are each one-half the total load wl; the moment at a distance x from the left support is

$$\frac{wlx}{2} - \frac{wx^2}{2},$$

and the differential equation becomes:

$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}. \tag{1}$$

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + C_1. \tag{2}$$

From symmetry it is evident that the maximum deflection is at the middle

$$\frac{dy}{dx} = 0 \text{ when } x = \frac{l}{2};$$

$$C_1 = -\frac{wl^3}{24}. \tag{3}$$

$\left(\frac{C_1}{EI}\right.$ is the slope of the elastic line at the left support.$\left.\right)$

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}. \tag{4}$$

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24} + C_2. \tag{5}$$

At $x = 0$, $y = 0$; $C_2 = 0$; (6)

$$EIy = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3x}{24}. \tag{7}$$

When $x = \frac{l}{2}$ the deflection is a maximum,

$$EIy_{\max} = \frac{wl^4}{12}\left(\frac{1}{8} - \frac{1}{32} - \frac{1}{4}\right) = -\frac{5wl^4}{384};$$

$$y_{\max} = -\frac{5wl^4}{384EI} = -\frac{5Wl^3}{384EI}. \quad \text{Formula XVIII.}$$

Substituting $x = l$ in Equation (7), the deflection at the right support is found to be zero. This condition might have been used to determine one of the constants.

This beam might be regarded as fixed at the middle where the slope is zero, and to consist of two cantilevers of length $\frac{l}{2}$ which are bent downward by the distributed load and bent upward by the end reactions. The deflection at one end is:

$$\text{downward,} \quad \frac{\frac{W}{2}\left(\frac{l}{2}\right)^3}{8EI} = \frac{Wl^3}{128EI},$$

$$\text{upward,} \quad \frac{\frac{W}{2}\left(\frac{l}{2}\right)^3}{3EI} = \frac{Wl^3}{48EI};$$

$$\text{total deflection upward,} \quad \frac{Wl^3}{384EI}(8 - 3) = \frac{5Wl^3}{384EI}.$$

The deflection at any point, measured upward from the middle, may be calculated in a similar way.

Problems

1. A 2-inch by 12-inch floor joist is 15 feet long between supports. If E is 1,200,000 pounds per square inch, find the deflection which is caused by a load of 90 pounds per foot. *Ans.* 0.296 inch.
2. A 24-inch, 80-pound I-beam supports the middle of the floor of a room which is 30 feet square. The total floor load is 100 pounds per square foot. If E is 29,000,000 pounds per square inch, find the deflection at the middle of the beam. Find the fiber stress at the dangerous section.
3. An 8-inch, 18-pound I-beam rests on two supports which are 10 feet apart and overhangs one support 5 feet. When a load of 12,000 pounds is uniformly distributed over the portion between the supports, how much is the free end elevated and how much is the point midway between the supports depressed? *Ans.* 0.262 inch; 0.164 inch.
4. A beam of length l and depth d has its neutral surface midway between the top and bottom. The beam is supported at the ends and carries a uniformly distributed load which makes the maximum stress equal to S. Find the deflection at the middle.

$$\textit{Ans.}\ \text{Deflection} = \frac{5Sl^2}{24Ed}.$$

5. If E is 29,000,000 and S is 16,000 pounds per square inch, find the maximum allowable deflection in a 12-inch I-beam which is 15 feet long.
6. What is the equation of the elastic line in terms of k, if $x = kl$?

$$\textit{Ans.}\ y = -\frac{wl^4k}{24EI}(1 - 2k^2 + k^3) = -\frac{Wl^3k}{24EI}(1 - 2k^2 + k^3).$$

78. Beam Supported at the Ends with Load at the Middle by Double Integration.—If P is the load at the middle, each reaction is $\frac{P}{2}$, and the moment from the left end to the middle is $\frac{Px}{2}$. (See Fig. 77.) For the portion of the beam between the left end and the middle.

$$EI\frac{d^2y}{dx^2} = \frac{Px}{2}. \tag{1}$$

$$EI\frac{dy}{dx} = \frac{Px^2}{4} + C_1. \tag{2}$$

At the middle, from the symmetry of the sides, $\frac{dy}{dx} = 0$;

$$C_1 = -\frac{Pl^2}{16}. \tag{3}$$

$$EI\frac{dy}{dx} = \frac{Px^2}{4} - \frac{Pl^2}{16}. \tag{4}$$

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16} + C_2. \tag{5}$$

At the left support, where $x = 0$, $y = 0$:

$$C_2 = 0. \tag{6}$$

$$EIy = \frac{Px^3}{12} - \frac{Pl^2x}{16}. \tag{7}$$

At the middle, where $x = \frac{l}{2}$.

$$y_{\max} = \frac{Pl^3}{96EI} - \frac{Pl^3}{32EI} = -\frac{Pl^3}{48EI}. \qquad \text{Formula XIX.}$$

Since the moment equation applies only to the left half of the beam, the formulas derived from this equation are not valid beyond the middle. From the middle to the right end, the moment is $\frac{P}{2}(l - x)$.

$$EI\frac{d^2y}{dx^2} = \frac{P}{2}(l - x). \tag{8}$$

$$EI\frac{dy}{dx} = -\frac{P(l - x)^2}{4} + \left[C_3 = \frac{Pl^2}{16}\right]. \tag{9}$$

$$EIy = \frac{P(l - x)^3}{12} + \frac{Pl^2x}{16} + C_4. \tag{10}$$

At the right support, where $x = l$, $y = 0$; $C_4 = -\frac{Pl^3}{16}$.

$$EIy = \frac{P(l - x)^3}{12} + \frac{Pl^2x}{16} - \frac{Pl^3}{16}. \tag{11}$$

A beam supported at the ends and loaded at the middle may be regarded as two cantilevers, each of length $\frac{l}{2}$, which are bent upward by reactions $\frac{P}{2}$.

Formula XIX is largely used in the determination of the modulus of elasticity.

Problems

1. Find the slope at the left end and the slope at the right end and compare the results.
2. Find the deflection at one-fourth the length from the left end and at three-fourths the length from the left end.

   Ans. $y = -\frac{11Pl^3}{768EI}$.
3. Find the deflection at one-third the length and at two-thirds the length from the left end.

4. A spruce beam tested at the Bureau of Standards was $1\frac{3}{4}$ inches square. The beam was supported at points 24 inches apart and was loaded at the middle. When the load changed from 30 pounds to 208 pounds, the deflections at the middle increased 0.0472 inch. Find E.

Ans. E = 1,390,000 pounds per square inch.

5. A second spruce beam was 1.75 inches by 1.78 inches. The span was 24 inches and the load was applied at the middle. The deflection at the middle was measured by a dial which read in ten-thousandths of an inch. When the load was 31 pounds the dial reading (not the deflection) was 0.0314 inch. At loads of 251, 471, 691 and 911 pounds, respectively, the dial readings were 0.0839, 0.1342, 0.1882, and 0.2700 inch. The beam broke under a load of 973 pounds. Find E and the modulus of rupture.

6. A 15-inch, 50-pound I-beam rests on two supports 20 feet apart and overhangs the left support 6 feet. Find the deflection at the middle of the span and the elevation of the free end when a load of 6,000 pounds is placed midway between the supports, if E is 29,000,000 pounds per square inch. *Ans.* −0.1233 inch; 0.1102 inch.

7. If S is the allowable unit stress in a beam, d is the depth, and l is the length, and if the neutral surface is midway between the top and bottom, what is the expression for the maximum allowable deflection when the beam is loaded at the middle. By means of this formula find the maximum allowable deflection in a steel beam 20 feet long and 18 inches deep.

8. A selected beam of red oak, 1.75 inches wide and 1.25 inches deep, was placed on two supports 12 inches apart and a load applied at the middle. When an addition of 607 pounds was made to the load, the deflection at the middle was increased 0.050 inch. Find E.

Ans. 1,534,000 pounds per square inch.

9. In the beam of Problem 8 an addition of 721 pounds produced a deflection of 0.059 inch. Find E.

10. In Problem 8 how much would the last significant figures of the value of E be changed if the deflection readings were incorrect 0.0005 inch? If the breadth were incorrect 0.005 inch? If the depth were incorrect 0.005 inch? If the load were incorrect 1 pound? How much would E be changed if all these errors occurred at once in the same direction?

Ans. An error of 0.005 inch in the depth would change the result 1.2 per cent., a change of 18 in the significant figures.

11. The beam of Problem 8 broke under a load of 2,315 pounds. Find the fiber stress at rupture.

79. Beam with Constant Moment, by Double Integration.—Figure 110 shows a beam which is supported at two points at a distance l apart, overhangs each support, and carries a load on each end. The moment at the left support is $-Pa$ and the moment at the right support is $-Qb$. If $Pa = Qb$, the left reaction is equal to P and the right reaction is equal to Q. The loads and reactions then form two equal and opposite couples, and the

moment in the beam between the supports is constant and is equal to $-Pa$ (or $-Qb$). With the origin of coördinates at the left support,

$$EI\frac{d^2y}{dx^2} = -Pa; \tag{1}$$

$$EI\frac{dy}{dx} = -Pax + C_1; \tag{2}$$

$$EIy = -\frac{Pax^2}{2} + C_1x + C_2. \tag{3}$$

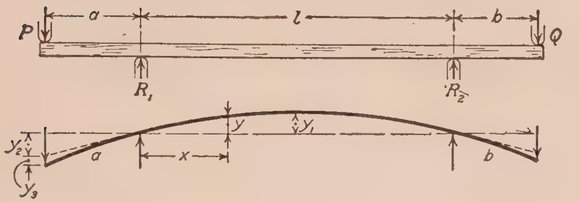


FIG. 110.

At the left support, where $x = 0$, $y = 0$, hence $C_2' = 0$.
At the right support, where $x = l$, $y = 0$.

$$0 = -\frac{Pal^2}{2} + C_1l,$$

$$C_1 = \frac{Pal}{2}.$$

When this value of C_1 is substituted in Equation (2).

$$EI\frac{dy}{dx} = -Pax + \frac{Pal}{2}. \tag{4}$$

At the point of maximum deflection, $\frac{dy}{dx} = 0$ and $x = \frac{l}{2}$. The maximum deflection is at the middle of the span.

The equation of the elastic line is

$$y = \frac{Pa}{2EI}(lx - x^2) = \frac{Pax}{2EI}(l - x). \tag{5}$$

$$y_{\max} = \frac{Pal^2}{8EI}. \tag{6}$$

If P and Q are equal and a and b are equal, it is evident that the elastic line is symmetrical with respect to the middle of the span. Under these conditions, it could be assumed that $\frac{dy}{dx}$ is

zero when x is $\frac{l}{2}$, and the value of C_1 could be obtained from the first integral. If P and Q are not equal, but the product Pa equals Qb, the symmetry is not self-evident, and it is better to determine both constants from the second integral.

Problems

1. A board, 6 inches wide and 1 inch thick, rests on two supports 80 inches apart. A load of 30 pounds is placed 16 inches to the left of the left support and a load of 20 pounds is placed 24 inches to the right of the right support. If E is 1,200,000 pounds per square inch, what is the deflection upward midway between the supports? What is the slope at each support?

   Ans. $y = 0.64$ inch; $\frac{dy}{dx} = 0.032$.

2. In Problem 1, find the deflection at 20 inches to the right of the left support.
3. In Problem 1, find the radius of curvature of the portion of the beam between the supports and calculate the deflection at the middle and the slope over the supports geometrically.

The overhanging ends of Fig. 110 are cantilevers. The deflection of each load *from the tangent at the support* (y_3 of Fig. 110) is given by Formula XVI. The deflection of the tangent at the support from the horizontal line through the supports (y_2 of Fig. 110) is the distance from the load to the support multiplied by the slope at the support.

$$y_2 + y_3 = -\frac{Pa^2l}{2EI} - \frac{Pa^3}{3EI} = -\frac{Pa^2}{6EI}(3l + 2a). \qquad (7)$$

Problems

4. In Problem 1, find the deflection of the load of 30 pounds downward from the horizontal line through the supports.

   Ans. $0.512 + 0.068 = 0.580$ inch.

5. A beam of total length l is supported at two points, each of which is $\frac{l}{4}$ from one end and carries a load of $\frac{P}{2}$ at each end. How much is the middle deflected upward and how much is each end deflected downward from the supports?

Figure 111 shows a method of loading and supporting a beam which is considerably used in testing. The beam is supported at the ends and two equal loads are applied at equal distances

from the ends. If the weight of the beam is neglected, the shear is zero and the moment is constant between the loads. The fiber stress is the same for the entire distance between the loads. If the beam is uniformly strong, it may fail at any point between

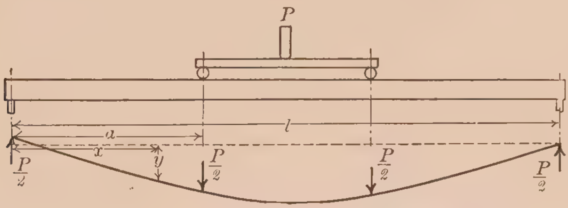


FIG. 111.—Supports at ends, two loads symmetrically placed.

the loads. If any section in this portion is weaker than the remainder, failure will occur at this weaker section. If the beam were loaded at the middle, and the weakest section were some distance from the middle, it would show considerably greater strength than if it were loaded as in Fig. 111. Figure 111 is equivalent to Fig. 110 inverted.

If l is the total length in Fig. 111, and a is the distance from each end to the nearest load, the length under constant moment is $l - 2a$. This distance must be used in place of l in the preceding equations.

Problems

6. A beam of length l is supported at the ends and carries a load $\frac{P}{2}$ at a distance a from each support (Fig. 111). How much is the middle of the beam deflected downward below the horizontal line through the loads? How much are the loads deflected downward below the supports? What is the total deflection of the middle below the supports?

$$Ans.\ \frac{Pa(l-2a)^2}{16EI};\ \frac{Pa^2(3l-4a)}{12EI};\ \text{total deflection, } \frac{Pa(3l^2-4a^2)}{48EI}.$$

7. In Problem 6, find the total deflection at the middle when the loads are at the *third-points*, that is, when a is one-third of l.

$$Ans.\ y_{\max} = -\frac{23Pl^3}{1{,}296EI}.$$

8. Solve Problem 7 if the loads are at the *fourth-points.*

$$Ans.\ y_{\max} = -\frac{11Pl^3}{768EI}.$$

9. A 7-inch by 16-inch beam of short-leaf yellow pine, tested by Professor A. N. Talbot, was placed on two supports, each 13 feet 6 inches apart,

and loaded at points 4 feet 6 inches from the supports. When the total load was 40,000 pounds, the deflection of the middle below the supports was 0.825 inch. Find the modulus of elasticity and the maximum unit stress. *Ans.* $E = 1{,}531{,}000$ pounds per square inch.*

10. Draw a sketch of the loaded beam of Problem 10 accurately to scale. What was the radius of curvature between the loads?

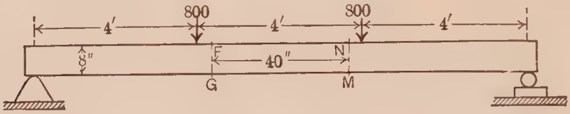


FIG. 112.—Beam loaded at third points.

11. A 6-inch by 8-inch wooden beam supported at points 12 feet apart is loaded with two equal loads of 800 pounds each 4 feet from the supports. If E is 1,200,000, what is the deflection under a load and at the middle? *Ans.* Under a load, 0.240 inch; at middle, 0.276 inch.

12. In Problem 11, two vertical lines are ruled on one side of the beam 20 inches on the right and left of the middle. When the load is applied, what angles will these lines make with each other?

13. In Problem 12, Fig. 112, the distance FN between the upper ends of the lines is measured with a delicate extensometer. How much is this distance diminished when the loads are applied, and what is the unit deformation in 40 inches? *Ans.* Total, 0.0200 inch; unit, 0.00050 inch.

80. Beam Supported at the Ends and Loaded at Any Point, by Double Integration.—The moment equation changes at the concentrated load, consequently two differential equations must be solved to obtain the equation of the elastic line for the entire beam. It is not possible to obtain the equation of the elastic line for the portion between one support and the load, without solving the second differential equation. The slope of the tangent is not zero at the middle nor under the load. If the differential equation were integrated for the portion from A to G of Fig. 113, there would be only one known condition ($y = 0$, when $x = 0$) for the determination of the two integration constants. If the right portion were integrated alone, there would be the single condition that $y = 0$, when $x = l$. When both equations are integrated together, there are two additional conditions. Under the load where $x = a$ (Fig. 113) the slope is the same and the deflection is the same for both sets of equa-

* From Bulletin No. 41 of the University of Illinois Engineering Experiment Station. The deflection as given was scaled from the load-deformation diagram.

tions. These with the two conditions already mentioned make it possible to evaluate the four integration constants.

Figure 113 represents a beam of length l supported at the ends with a load P at a distance a from the left end. If $l - a = b$,

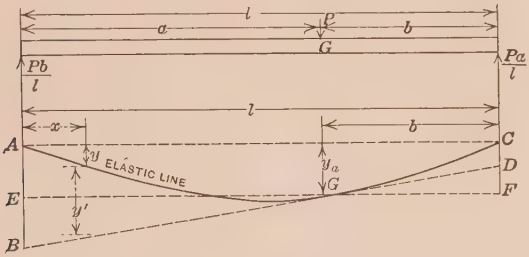


FIG. 113.—Beam with load at any point.

the left reaction is $\frac{Pb}{l}$. For the left portion of the beam

$$M = \frac{Pbx}{l},$$

and to the right of the load

$$M = \frac{Pbx}{l} - P(x - a).$$

For all points from $x = 0$ *to* $x = a$, *inclusive,*	*For all points from* $x = a$ *to* $x = l$, *inclusive,*
$EI\frac{d^2y}{dx^2} = \frac{Pbx}{l}.$ (1)	$EI\frac{d^2y}{dx^2} = \frac{Pbx}{l} - P(x - a).$ (2)
$EI\frac{dy}{dx} = \frac{Pbx^2}{2l} + C_1.$ (3)	$EI\frac{dy}{dx} = \frac{Pbx^2}{2l} - \frac{P(x - a)^2}{2} + C_3.$ (4)

The curve is continuous under the load with no abrupt change of slope. When $x = a$ the value of $\frac{dy}{dx}$ calculated from (3) is the same as when calculated from (4). This makes the first members of the two equations equal and, consequently, the second members are equal when a is substituted for x.

$$\frac{Pba^2}{2l} + C_1 = \frac{Pba^2}{2l} - \frac{P(a - a)^2}{2} + C_3;$$

$$C_1 = C_3.$$

Substituting C_1 for C_3 in Equation (4) and integrating both equations,

$$EIy = \frac{Pbx^3}{6l} + C_1x + C_2. \quad (5)$$

When $x = 0$, $y = 0$,
hence $C_2 = 0$.

$$EIy = \frac{Pbx^3}{6l} - \frac{P(x-a)^3}{6} + C_1x + C_4. \quad (6)$$

When $x = a$ the values of y from (5) and (6) are the same and the second members of these equations are equal, from which:

$$0 = C_2 = C_4.$$

When $x = l$ in (6), $y = 0$;

$$C_1 = -\frac{Pbl^2}{6l} + \frac{P(l-a)^3}{6l} = -\frac{Pb}{6l}(l^2 - b^2). \quad (7)$$

Substituting the value of C_1 from (7) in (5),

$$EIy = \frac{Pbx^3}{6l} - \frac{Pb(l^2 - b^2)x}{6l}. \quad (8)$$

Substituting C_1 in (3) and equating to zero,

$$x^2 = \frac{l^2 - b^2}{3} = \frac{a(a+2b)}{3}, \quad (9)$$

gives the point of maximum deflection, provided b is less than a. Substituting x from (9) in (5),

$$y_{\max} = -\frac{Pb(l^2 - b^2)\sqrt{3(l^2 - b^2)}}{27EIl} = -\frac{Pba(a+2b)\sqrt{3a(a+2b)}}{27EIl}. \quad (10)$$

The deflection under the load is

$$y = -\frac{Pa^2b^2}{3EIl}. \quad (11)$$

Problems

1. Show that the point of maximum deflection is never beyond $\frac{l}{\sqrt{3}}$.

2. A 3-inch by 2-inch rectangular beam, 10 feet long, supports a load of 45 pounds 6 feet from the left end. Find the deflection at the middle, under the load, and at the point of maximum deflection if the beam is supported at the ends and E is 1,500,000 pounds per square inch.
Ans. At middle, 0.510 inch; under load, 0.498 inch; maximum, 0.512 inch.

3. Find the deflection at the middle of a beam of length l which carries a load P at a distance b from one end, if b is less than one-half the length.

Ans. $y = -\frac{Pb}{48EI}(3l^2 - 4b^2)$.

4. A beam of length l carries a load P at the middle. Find the deflection at a distance b from the right end by means of the Equations of Article 78. From this result and the proposition of Article 75, solve Problem 3.

5. Find the deflection at the middle when $b = \frac{l}{3}$. Why is the result the same as the answer of Problem 7 of Article 79?

81. The Method of Area Moments.—The calculation of deflections by the method of *area moments** has decided advantages in some cases, especially when the loads are concentrated and the slope is known at some point.

From the equation $M = EI\frac{d\theta}{dl} = EI\frac{d\theta}{dx}$, when θ is a small angle

$$d\theta = \frac{M}{EI}\,dx; \tag{1}$$

$$\theta = \int\frac{M}{EI}\,dx + C = \frac{1}{EI}\int M dx + C, \tag{2}$$

when I is constant.

Since $\int M dx$ is the area of the moment diagram, *the difference in slope between two points on a beam of uniform section is the area of the moment diagram between these points divided by EI.* If the moment of inertia varies, the difference of slope is the area of the $\frac{M}{I}$ diagram divided by E.

In Fig. 114, A_0B_0 represents a portion of a beam. The line $A_1B_1B_0$ which is straight from A_1 to B_1 and curved from B_1 to B_0 represents the same beam after the portion from B_1 to B_0 has been bent by a moment M. If the short portion between B_1 and B_2 is now bent by a moment M, the point A_1 moves to A_2. The angle between the tangent A_1B_1 and the tangent A_2B_2 (or the angle between the normals at B_1 and B_2) is $d\theta$. (It is assumed that the angles are so small that the angle θ is practi-

* This method was devised by Mohr and independently in America by Prof. Charles E. Greene, who began to teach it in 1873. See paper by A. E. GREENE in the *Michigan Technic* of June, 1910.

cally equivalent to tangent of θ, and the points $A_0A_1A_2$ lie on a straight line which is approximately vertical.) The deflection dy_a, from A_1 to A_2, when the infinitesimal length B_1B_2 is bent, is $A_2B_2\ d\theta$ (or $A_1B_1\ d\theta$, since B_1B_2 is infinitesimal). Since θ is

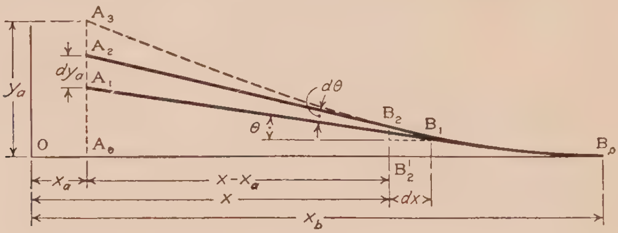


FIG. 114.

small, A_2B_2 is practically equivalent to its horizontal projection $A_0B'_2$, the length of which is $x - x_a$

$$dy_a = (x - x_a)d\theta; \tag{3}$$

$$dy_a = \frac{M}{EI}(x - x_a)dx. \tag{4}$$

The total deflection of the point A is

$$y_a = \int \frac{M}{EI}(x - x_a)dx. \qquad \text{Formula XX}$$

For a uniform beam

$$EIy_a = \int M(x - x_a)\,dx. \qquad \text{Formula XXI}$$

When the origin of coördinates as taken at the point A_0.

$$EIy_a = \int Mxdx \qquad \text{Formula XXII}$$

Since Mdx is an element of the moment diagram, $(x - x_a)$ Mdx is the moment of this element with respect to the line $A_0A_1A_2$. The integrals of Formulas XXI and XXII give the moment with respect to A of the entire moment diagram from A to B_o. *The distance y_a is the deflection of the point A from the tangent at B_o.* This is called the *Area Moments Method, the Slope Deflection Method*, or the *Integral of Mxdx Method.*

For a concentrated load or reaction, the moment diagram is a triangle; for a uniformly distributed load it is a parabola. Since the area and location of the center of gravity of these figures are known, the moment is usually computed geometrically

without integration. For other loadings it may be necessary to integrate for the moment. When the moment of inertia is not constant, and Formula XX is required to find the moment of the $\frac{M}{EI}$ or the moment of the $\frac{M}{I}$ diagram, it is generally advisable to integrate.

82. Cantilever Loaded at Free End, by Area Moments.—Figure 115 shows a cantilever with a load on the free end. The

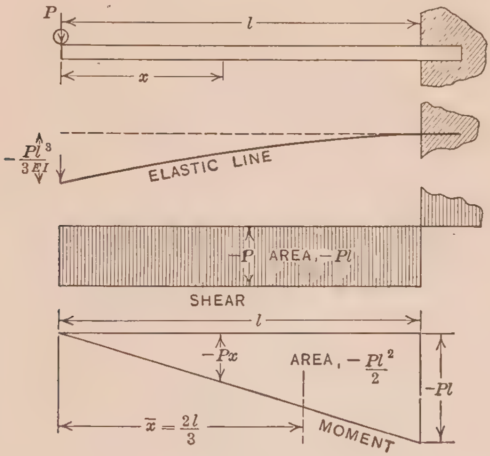


FIG. 115.—Area moments for cantilever.

figure also shows the elastic line with the deflections magnified, the shear diagram, and the moment diagram. The moment diagram is a negative triangle. The maximum ordinate is $-Pl$, which is the area of the shear diagram. The area of the moment triangle is $-\frac{Pl^2}{2}$ and its center of gravity is $\frac{2l}{3}$ from the left end. The deflection of the left end from the horizontal tangent at the right end is given by

$$EIy_{\max} = -\frac{Pl^2}{2} \times \frac{2l}{3} = -\frac{Pl^3}{3}; \qquad (1)$$

$$y_{\max} = -\frac{Pl^3}{3EI}. \qquad \text{Formula XVI.}$$

The change in slope from the free end to the fixed end is the area of the moment diagram divided by EI. Since the beam is horizontal at the fixed end, the slope at the free end is

$$\theta - \frac{Pl^2}{2EI} = 0, \tag{2}$$

$$\theta = \frac{Pl^2}{2EI}. \tag{3}$$

(Compare with Article 73.)

Figure 116 is the moment diagram used to find the deflection at a distance x from the free end. The area to the right of the

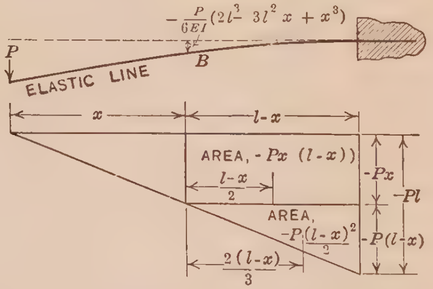


FIG. 116.—Area moments for any point of cantilever.

point B may be broken up into the rectangle of base $l - x$ and altitude $-Px$, and the lower triangle of base $l - x$ and altitude $-P(l - x)$. The moment arm of the rectangle is $\frac{l - x}{2}$ and that of the triangle is $\frac{2(l - x)}{3}$.

$$\text{Moment of rectangle} = -Px(l - x) \times \frac{l - x}{2} = -\frac{Px}{2}(l - x)^2. \tag{4}$$

$$\text{Moment of triangle} = -\frac{P(l - x)^2}{2} \times \frac{2(l - x)}{3} = -\frac{P(l - x)}{3}(l - x)^2. \tag{5}$$

$$\text{Total moment} = EIy = -\frac{Pl^3}{3} + \frac{Pl^2x}{2} - \frac{Px^3}{6}. \tag{6}$$

$$y = -\frac{P}{6EI}(2l^3 - 3l^2x + x^3). \tag{7}$$

Instead of the *sum* of the moments of the triangle and the rectangle to the right of B, the *difference* of the moments of the entire triangle and the small triangle to the left of B may be

used. The area of the large triangle is $-\frac{Pl^2}{2}$, and its center of gravity is $\frac{2l}{3} - x$ to the *right* of B. The area of the small triangle is $-\frac{Px^2}{2}$ and its center of gravity is $\frac{x}{3}$ to the *left* of B.

$$-\frac{Pl^2}{2} \times \left(\frac{2l}{3} - x\right) = -\frac{Pl^3}{3} + \frac{Pl^2x}{2}; \tag{8}$$

$$-\left(-\frac{Px^2}{2}\right) \times \left(-\frac{x}{3}\right) = -\frac{Px^3}{6}; \tag{9}$$

$$y = -\frac{P}{6EI}(2l^3 - 3l^2x + x^3). \tag{7}$$

Problems

1. Find the slope at a distance x from the free end by means of the area of the moment diagram and compare with Article 73.

   Ans. $EI\theta = \frac{Pl^2}{2} - \frac{Px^2}{2}$.

2. A beam of length l is fixed at the right end and carries a load P on the left end. Find the deflection at one-third the length from the left end by means of the moments of the two moment triangles without the use of Equation (7). Check by Equation (7).
3. A 4-inch by 6-inch cantilever, 10 feet long, is deflected 1.2 inches at the end by a load of 180 pounds on the end. Find the modulus of elasticity.

   Ans. E = 1,200,000 pounds per square inch.

4. A 12-inch, 31.5-pound I-beam is used as a cantilever. A load of 4,000 pounds is placed 8 feet 4 inches from the fixed end. If E is 29,000,000 pounds per square inch, what is the deflection under the load, and what is the maximum fiber stress?

   Ans. Deflection = 0.213 inch.

5. A 15-inch I-beam as a cantilever 10 feet long carries a load on the free end which makes the maximum fiber stress 14,000 pounds per square inch. If E is 29,000,000 pounds per square inch, what is the deflection under the load?
6. Differentiate Equation (7) to find the slope of the tangent to the elastic line. Compare with the answer of Problem 1.

83. Cantilever Loaded at Any Point, by Area Moments.—Figure 117 shows a cantilever of length l, which carries a load P at a distance a from the free end. The area of the moment triangle is $-\frac{P(l-a)^2}{2}$. The moment arm for finding the deflection at the free end is $a + \frac{2(l-a)}{3}$, which reduces to $\frac{2l + a}{3}$.

$$EIy = -\frac{P(l-a)^2}{2} \times \frac{2l+a}{3} = -\frac{P}{6}(2l^3 - 3l^2a + a^3); \quad (1)$$

$$y_{max} = -\frac{P}{6EI}(2l^3 - 3l^2a + a^3). \quad (2)$$

If x is substituted for a, Equation (2) becomes identical with Equation (7) of the preceding article. This identity is an illustration of the principle of Article 75.

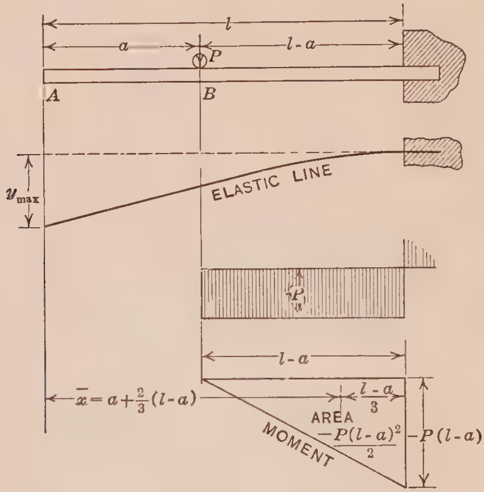


Fig. 117.—Cantilever with load at any point.

The slope under the load, or at any point to the left of the load, is equal to the area of the moment diagram with the positive sign divided by EI. This slope multiplied by a gives the vertical distance of the end of the beam below the load.

Problem

1. A 4-inch by 4-inch cantilever, 15 feet long, carries a load of 40 pounds 5 feet from the free end and a load of 60 pounds 10 feet from the free end. Find the deflection at the free end if E is 1,350,000 pounds per square inch. Solve from the two moment triangles without using Equation (2).

Ans. $y_{max} = 2$ inches.

84. Cantilever with Uniformly Distributed Load, by Area Moments.—Figure 118 shows a cantilever with a uniformly distributed load of w per unit length. The shear diagram is a triangle of maximum altitude $-wl$ and area $-\frac{wl^2}{2}$. The moment

diagram is a parabola. The maximum ordinate, which is the area of the shear diagram, is $-\frac{wl^2}{2}$. The area of a parabola which is convex toward the base is one-third the product of the base by the altitude.

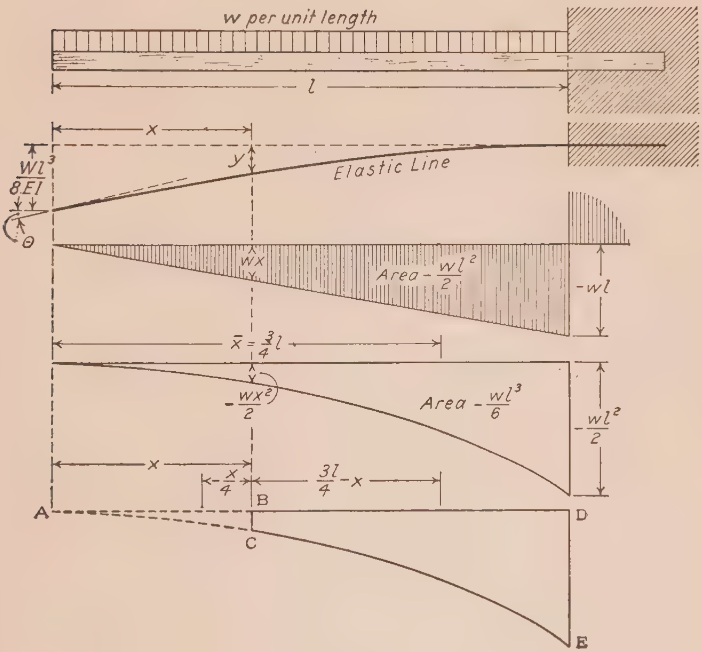


FIG. 118.

$$\text{Area of moment diagram} = -\frac{wl^2}{2} \times \frac{l}{3} = -\frac{wl^3}{6}. \tag{1}$$

Since the slope at the fixed end is zero, the slope at the free end is

$$\theta = \frac{wl^3}{6EI}. \quad \text{(Compare with Article 76).} \tag{2}$$

The distance of the center of gravity of this parabola from the vertex is $\frac{3l}{4}$.

See Fig. 230. The *area* of a plane figure whose height varies as the square of x is analogous to the *volume* of a pyramid, since the areas of the cross-sections of the pyramid vary as the square of the distance from the vertex.

$$EIy_{max} = -\frac{wl^3}{6} \times \frac{3l}{4} = -\frac{wl^4}{8}, \tag{3}$$

$$y_{max} = -\frac{wl^4}{8EI} = -\frac{Wl^3}{8EI}, \quad \text{Formula XVIII.}$$

in which $W = wl =$ the total distributed load.

To find the deflection at a point at a distance x from the free end, the moment of the parabola to the left of the point is subtracted from the moment of the entire diagram. The moment arm of the entire diagram ADE is $\frac{3l}{4} - x$. The area of the parabola ABC is $-\frac{wx^3}{6}$ and its moment arm with respect to B is $-\frac{x}{4}$

$$\left(-\frac{wl^3}{6}\right)\left(\frac{3l}{4} - x\right) = -\frac{wl^4}{8} + \frac{wl^3x}{6}; \tag{4}$$

$$-\left(-\frac{wx^3}{6}\right)\left(-\frac{x}{4}\right) = -\frac{wx^4}{24}; \tag{5}$$

$$EIy = -\frac{w}{24}(3l^4 - 4l^3x + x^4); \tag{6}$$

$$y = -\frac{w}{24EI}(3l^4 - 4l^3x + x^4), \tag{7}$$

which is the equation of the elastic line. (Compare Article 76.)

Problems

1. Find the deflection at the middle of a cantilever which is uniformly loaded. Solve by means of the moments of the areas without the use of Equation (7).

   Ans. $y = -\frac{17Wl^3}{384EI}$.

2. What is the slope at the middle of a cantilever with a uniformly distributed load?
3. A 4-inch by 6-inch wooden cantilever is 10 feet long and carries a distributed load of 48 pounds per foot. If E is 1,500,000, what is the deflection at the free end and what is the maximum unit stress?

   Ans. $y_{max} = 0.96$ inch; $S = 1{,}200$ pounds per square inch.

4. In Problem 3, what is the slope of the tangent at the free end and at the middle?

Figure 119 shows a cantilever which carries a distributed load over a portion of its length adjacent to the fixed end, while the remainder, of length a, is not loaded. To find the deflection

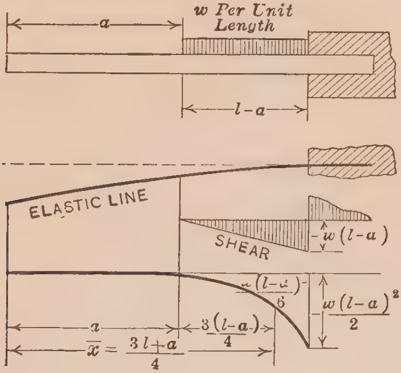


FIG. 119.—Cantilever with distributed load over part of length.

at the free end, the area of the moment diagram is $-\dfrac{w(l-a)^3}{6}$, and the moment arm with respect to the end is

$$a + \frac{3}{4}(l - a) = \frac{3l + a}{4}. \tag{8}$$

$$EIy_{\max} = -\frac{w(l-a)^3}{6} \times \frac{3l + a}{4}, \tag{9}$$

$$y_{\max} = -\frac{w}{24EI}(3l^4 - 8l^3a + 6l^2a^2 - a^4). \tag{10}$$

Problems

5. A cantilever of length l carries a load of w per unit length over the half of its length adjacent to the fixed end and no load over the remainder. Find the deflection at the free end by means of the diagram and check by Equation (10).

   Ans. $y_{\max} = -\dfrac{7wl^4}{384EI}$.

6. Find the deflection at the middle of the beam of Problem 5 by Formula XVIII and add to this the slope at the middle multiplied by one-half the length.

7. A cantilever of length l carries a load of w per unit length over the half adjoining the free end and no load over the remainder. Find the deflection at the free end.

   Ans. $y_{\max} = -\dfrac{41\,wl^4}{384EI}$.

8. Solve Problem 7 if the load extends over four-tenths of the length.

85. Beam Supported at the Ends and Loaded at the Middle, by Area Moments.—Figure 120 shows a beam which is supported at the ends and loaded at the middle. The positive moment of the left reaction is the triangle DEF. The negative moment

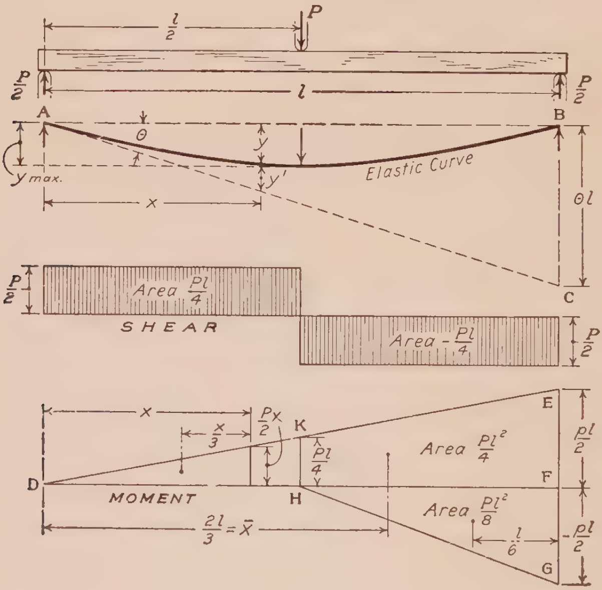


FIG. 120.

of the concentrated load is the triangle HFG. The combined moment to the right of the load is the positive triangle KFH. (The line KF is not shown.) It is generally best to use the positive and negative triangles separately instead of the resultant.

The deflection upward of the left support from the tangent at the middle is derived from the moment triangle DKH.

$$EIy = \frac{Pl^2}{16} \times \frac{l}{3} = \frac{Pl^3}{48}, \tag{1}$$

in which y is the deflection *upward* of the left end and is opposite in sign to the deflection of the middle *downward* below the line of the supports.

$$y_{\max} = -\frac{Pl^3}{48EI}. \qquad \text{Formula XIX.}$$

The deflection at any point may be obtained by means of the deflection upward of the point from the tangent at the

middle. It is more convenient, however, to use the deflection upward from the tangent at the left end. Since the slope is zero at the middle, the slope at the end is given by

$$EI\theta + \frac{Pl^2}{16} = 0 \tag{2}$$

$$\theta = -\frac{Pl^2}{16EI}. \tag{3}$$

At a distance x from the left end, the tangent at the left end is $-\frac{Pl^2x}{16EI}$ from the horizontal line through the supports. If y' is the deflection upward from the tangent,

$$EIy' = \frac{Px^2}{4} \times \frac{x}{3} = \frac{Px^3}{12}; \tag{4}$$

$$y = -\frac{Pl^2x}{16EI} + \frac{Px^3}{12EI}, \tag{5}$$

in which y is the deflection of any point to the left of the load below the line of the supports.

Problems

1. Find the deflection of the right end of the beam of Fig. 120 upward from the tangent at the left end, and show that it is equal to the slope at the left end multiplied by the length.

$$\textit{Ans.}\ EI \times CB = \frac{Pl^2}{4} \times \frac{l}{3} - \frac{Pl^2}{8} \times \frac{l}{6} = \frac{Pl^3}{16}.$$

2. Find the deflection upward of the left end from the tangent at the right end.
3. Find the slope of the line at a distance x from the left support and compare with Article 78.
4. Find the deflection at one-third the length from the left support directly from the moment triangle. Also find the deflection at two-thirds the length from the left end by means of the triangles.

86. Beam Supported at the Ends and Uniformly Loaded, by Area Moments.—The moment at a distance x from the left support is $\frac{wlx}{2} - \frac{wx^2}{2}$. These terms are shown separately in the lower diagram of Fig. 121. The positive triangle represents the first term and the negative parabola represents the second term. The second term is that of a cantilever with a uniformly distributed load, and the first term is that of a cantilever with a concentrated load at the left end, *pushing upward.* From the

symmetry, it is evident that the beam is horizontal at the middle, and the deflection at any point upward from the tangent at the middle may be calculated by combining the two cases of the cantilever with the proper signs.

If the distance $\frac{wx^2}{2}$ is measured downward from the line $\frac{wlx}{2}$,

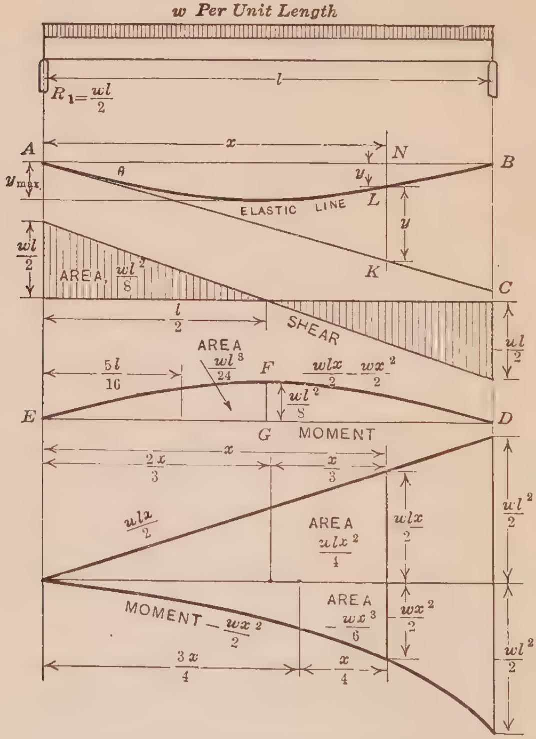


FIG. 121.—Beam supported at ends, distributed load.

the remainder gives the parabola *EFD*, which represents the entire moment. In some cases it is convenient to use this single parabola in calculating the deflection, generally, however, it is better to use the separate figures of the lower diagram.

The deflection of the left end upward from the tangent at the middle may be calculated by means of the parabola *EFD*. This parabola is *concave* toward the base. Its area is two-thirds

the product of the base by the altitude and the center of gravity of the half EFG is five-eighths of EG from E. (See Table XXIV)

$$-EIy_{\max} = \frac{wl^3}{24} \times \frac{5l}{16} = \frac{5wl^4}{384}; \tag{1}$$

$$y_{\max} = -\frac{5wl^4}{384EI} = -\frac{5Wl^3}{384EI}. \quad \text{Formula XVIII.}$$

The slope of the tangent at the left end is found by the area of of the moment diagram. Since the slope at the middle is zero,

$$EI\theta + \frac{wl^3}{24} = 0, \tag{2}$$

$$\theta = -\frac{wl^3}{24EI}. \tag{3}$$

In Fig. 121, AC is the tangent to the elastic line at the left end. The distance NK from the line of the supports to a point K on this line is

$$NK = \theta x = -\frac{wl^3x}{24EI}. \tag{4}$$

The deflection of the elastic line upward from the tangent AC at a distance x from the left end is

$$EIy' = \frac{wlx^2}{4} \times \frac{x}{3} - \frac{wx^3}{6} \times \frac{x}{4} = \frac{wlx^3}{12} - \frac{wx^4}{24}; \tag{5}$$

$$y' = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI}. \tag{6}$$

The deflection y from the line of the supports is the algebraic sum of NK and y'

$$y = NK + y' = -\frac{wl^3x}{24EI} + \frac{wlx^3}{12EI} - \frac{wx^4}{24EI},$$

$$y = -\frac{wx}{24EI}(l^3 - 2lx^2 + x^3). \tag{7}$$

If $x = kl$, Equation (7) is

$$y = -\frac{wl^4k}{24EI}(1 - 2k^2 + k^3). \tag{8}$$

Problems

1. Find the slope of the beam at a distance x from the left support from the area of the moment diagram. Compare with Article 77.

2. Find the deflection at one-fourth the length from the left support from the moment diagram without the use of Equation (7).

3. Find the deflection of the left end from the tangent at the right end. Divide the result by the length to get the slope at the right end. Check.

4. Find the deflection of the left end from the tangent at the middle by means of the positive triangle and negative parabola of the lower moment diagram of Fig. 121.

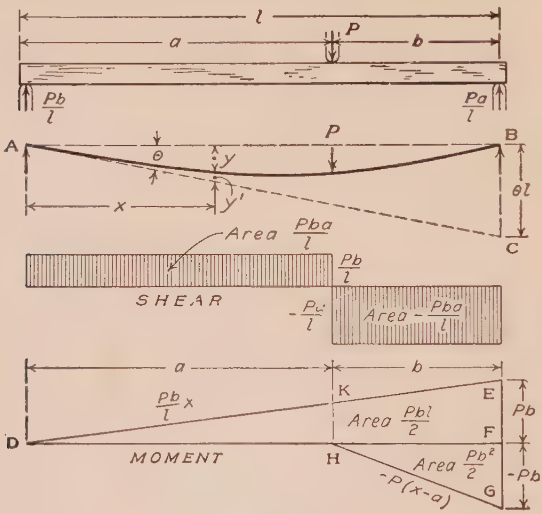


FIG. 122.

87. Beam Supported at the Ends with Load at Any Point, by Area Moments.—In Fig. 122, the moment diagram consists of a positive triangle of base l and a negative triangle of base b. The slope at the left end is found by dividing CB by the length.

$$EI \times CB = \frac{Pbl}{2} \times \frac{l}{3} - \frac{Pb^2}{2} \times \frac{b}{3} = \frac{Pb}{6}(l^2 - b^2) \qquad (1)$$

$$CB = \frac{Pb}{6EI}(l^2 - b^2); \qquad (2)$$

$$\theta = \tan\theta = \frac{CB}{l} = -\frac{Pb}{6EIl}(l^2 - b^2). \qquad (3)$$

The deflection y at any point is the algebraic sum of the deflection of the tangent at that point (which is negative) and deflection y' of the elastic line from the tangent.

Under the load,

$$EIy' = \frac{Pba^2}{2l} \times \frac{a}{3} = \frac{Pba^3}{6l} \qquad (4)$$

$$y = a\theta + y' = -\frac{Pba}{6EIl}(l^2 - b^2) + \frac{Pba^3}{6EIl}; \qquad (5)$$

$$y = -\frac{Pba}{6EIl}(l^2 - b^2 - a^2) = -\frac{Pa^2b^2}{3EIl}. \qquad (6)$$

At a distance x from the left support (if x is less than a),

$$EIy' = \frac{Pbx^2}{2l} \times \frac{x}{3} = \frac{Pbx^3}{6l}. \tag{7}$$

$$y = -\frac{Pbx}{6EIl}(l^2 - b^2) + \frac{Pbx^3}{6EIl}; \tag{8}$$

$$y = -\frac{Pbx}{6EIl}(l^2 - b^2 - x^2). \tag{9}$$

At the point of maximum deflection the slope is zero. Since the area of the moment diagram divided by EI measures the change in slope,

$$-\frac{Pb}{6EIl}(l^2 - b^2) + \frac{Pbx^2}{2EIl} = 0; \tag{10}$$

$$x^2 = \frac{l^2 - b^2}{3}; \quad x = \sqrt{\frac{l^2 - b^2}{3}}, \tag{11}$$

is the abscissa of the point of maximum deflection. When this value of x is substituted in Equation (8),

$$y_{\max} = -\frac{Pb(l^2 - b^2)\sqrt{3(l^2 - b^2)}}{27EIl} =$$

$$-\frac{Pba(a + 2b)\sqrt{3a(a + 2b)}}{27EIl}. \tag{12}$$

Problems

1. Find the position of maximum deflection by differentiating Equation (9).
2. Find the intercept of the tangent at the right end on the vertical line through the left end. From this find the slope of the tangent. Check by means of the area of the moment diagram.
3. Find the deflection at the middle.

88. Beam with Loads Symmetrically Placed, by Area Moments.—Figure 123 shows a beam which is supported at the ends and carries a load of $\frac{P}{2}$ pounds at a distance a from the left end and an equal load at the same distance from the right end. The moment diagram consists of two triangles and a rectangle. The deflection of the left end from the tangent at the middle is given by

$$-EIy_{\max} = \frac{Pa^2}{4} \times \frac{2a}{3} + \frac{Pa}{4}(l - 2a)\frac{l + 2a}{4}; \tag{1}$$

$$y_{\max} = -\frac{Pa}{48EI}(3l^2 - 4a^2). \tag{2}$$

The deflection of the point of application of a load above the tangent at the middle is

$$EIy' = \frac{Pa}{4}(l - 2a) \times \frac{l - 2a}{4} = \frac{Pa}{16}(l^2 - 4al + 4a^2). \quad (3)$$

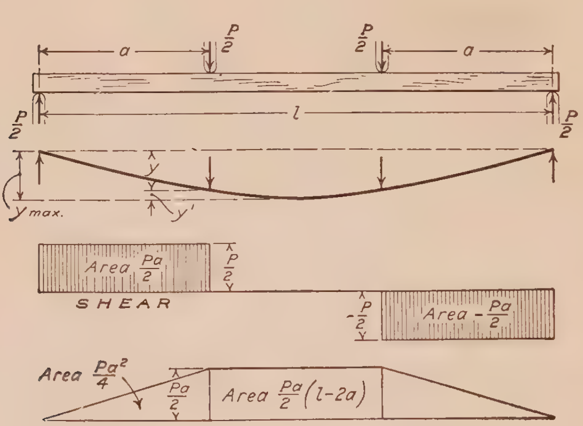


FIG. 123.

The deflection of the point of application of a load downward from the line through the supports is the algebraic sum of y_{max} and y';

$$y = -\frac{Pal^2}{16EI} + \frac{Pa^3}{12EI} + \frac{Pal^2}{16EI} - \frac{Pa^2l}{4EI} + \frac{Pa^3}{4EI}; \quad (4)$$

$$y = -\frac{Pa^2}{12EI}(3l - 4a). \quad (5)$$

Problems

1. Find the deflection under each load when a is one-half of l.
2. Find the slope of the beam at the end, and under a load.

89. Area Moments by Integration.—The beams which so far have been considered have all been of uniform cross-section and constant moment of inertia. When the moment of inertia varies, the expression for I cannot precede the integral sign in Formula XXI but this equation must be written

$$Ey = \int \frac{M}{I}(x - x_a)dx. \quad (1)$$

The moment of the $\frac{M}{I}$ *diagram is equal to the deflection multiplied by the modulus of elasticity.*

For a few forms of tapered beams with some particular loading, the $\frac{M}{I}$ diagrams are simple geometrical figures. For most beams and loadings, however, the diagrams are such that their moments must be determined by integration. Several illustrations of these integrations are given in Chapter XI.

When a beam of *uniform* section carries loads which are neither uniformly distributed nor concentrated at a few points, the moment diagram is usually of such form that it is advisable to integrate to determine its moment.

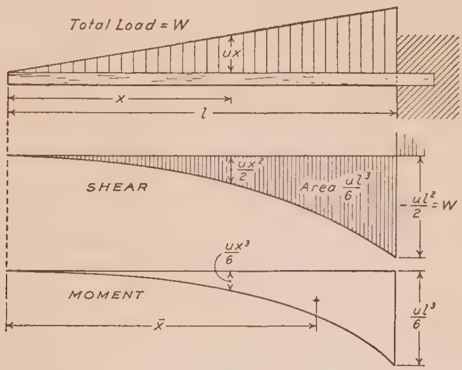


FIG. 124.

90. Cantilever with Load Increasing Uniformly from the Free End to the Fixed End.—Figure 124 shows a cantilever with a load which increases uniformly from the free end to the fixed end. The total load is W and the load per unit length at unit distance from the free end is u. At a distance x from the left end the load per unit length is ux, the shear is $-\frac{ux^2}{2}$, and the moment is $-\frac{ux^3}{6}$. The total load $W = \frac{ul^2}{2}$, which is the shear infinitely close to the fixed end.

To find the deflection at the free end,

$$EIy_{\max} = -\int \frac{ux^4}{6} dx = -\left[\frac{ux^5}{30}\right]_0^l = -\frac{ul^5}{30} = -\frac{Wl^3}{15}. \qquad (1)$$

$$y_{\max} = -\frac{Wl^3}{15EI}. \quad \text{(See Carnegie.)} \qquad (2)$$

To find the deflection at a distance a from the free end,

$$EIy = -\int \frac{ux^3}{6}(x - a)\,dx = -u\left[\frac{x^5}{30} - \frac{ax^4}{24}\right]_a^l \tag{3}$$

$$y = -\frac{u}{EI}\left(\frac{l^5}{30} - \frac{al^4}{24} + \frac{a^5}{120}\right) \tag{4}$$

Since a may be any value, it may be replaced by x

$$y = -\frac{W}{60EI}\left(4l^3 - 5l^2x + \frac{x^5}{l^2}\right) \tag{5}$$

Problems

1. Solve for Equation (2) by means of the area of the moment diagram. (See Table XXIV.)
2. Solve for the equation of the elastic line by double integration.

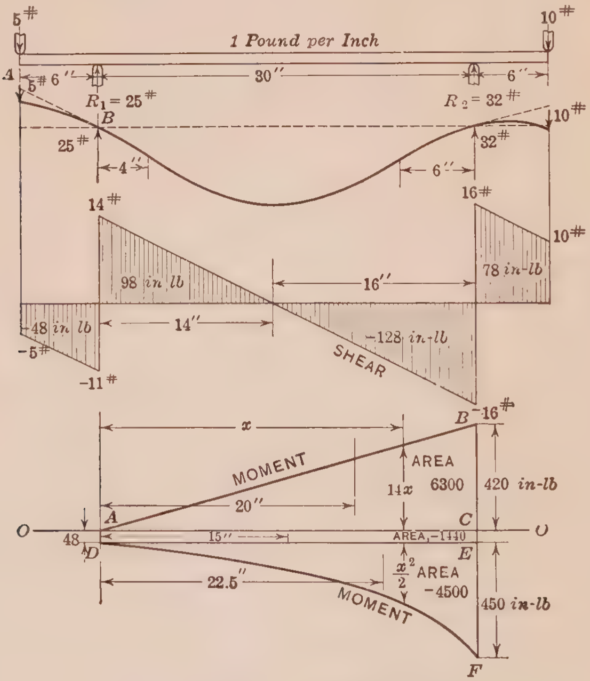


FIG. 125.—Beam overhanging supports.

91. Deflection by the General Moment Equation.—The general moment equation is frequently convenient for the solution of deflection problems. It is therefore advisable to interpret the equation geometrically for application of the method of area moments. Figure 125 shows a beam which overhangs the sup-

ports and carries a uniformly distributed load and a concentrated load on each end. The general moment equation may be applied to the portion between the supports with the origin of coördinates infinitely close to the right of the left support. The general moment equation, when there are no concentrated loads in the span under consideration, is

$$M = M_0 + V_0 x - \frac{wx^2}{2}. \tag{1}$$

In Fig. 125, $M_0 = -48$ and $V_0 = 14$. The first term of the moment equation is represented by a rectangle of altitude -48. The second term is represented by a positive triangle of maximum altitude 420. The third term is a negative parabola of maximum altitude 450. These are the characteristic figures for the general moment equation. If there were concentrated loads in this span, the moment of each load would be represented by a negative triangle.

To find the deflection of Fig. 125 by area moments, it is necessary to first find the slope at one support. The deflection of the right support upward from the tangent at the left support is

$$EIy' = 6{,}300 \times 10 - 1{,}440 \times 15 - 4{,}500 \times 7.5 = 7{,}650 \tag{2}$$

$$30\theta = \frac{-7{,}650}{EI};$$

$$\theta = \frac{-255}{EI},$$

in which θ is the angle which the tangent toward the right at B makes with the horizontal.

Problems

1. Find the deflection of the left support upward from the tangent at the right support and find the slope at the right support.

   Ans. $\theta = \dfrac{105}{EI}$ between the tangent to the right of the right support and the horizontal.

2. Check Problem 1 from the slope at the left support and the area of the moment diagram.
3. Find the deflection at the left end. First find the deflection from the tangent BA by Formulas XVI and XVII, then add the deflection of the point A from the horizontal.
4. Find the equation of the elastic line for the portion between the supports by area moments and check by double integration.

   Ans. $EIy = -255x - 24x^2 + \dfrac{7x^3}{3} - \dfrac{x^4}{24}.$

Figure 126 illustrates the application of the general moment equation to a cantilever with uniformly distributed load when it is desired to find the deflection at a distance x from the free end. Any horizontal distance from the free end between x and l may be represented by x'. (x' is not shown in Fig. 126.)

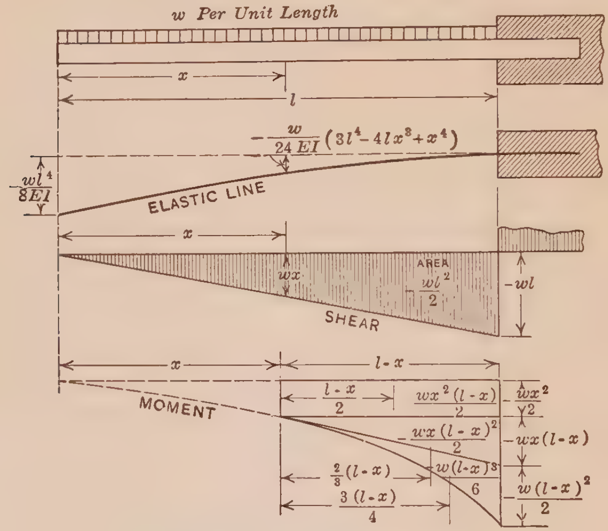


FIG. 126.—Cantilever with distributed load.

When the origin of coördinates for the general moment equation is taken at a distance x from the free end,

$$M = M_0 + V_0(x - x') - \frac{w(x - x')^2}{2}. \tag{3}$$

When $x' - x$ is considered a single variable, it is evident that the second term may be represented by a triangle and the third term may be represented by a parabola.

$$M_0 = -\frac{wx^2}{2} \text{ and } V_0 = -wx.$$

The area of the rectangle is $-\frac{wx^2(l - x)}{2}$; the area of the triangle is $-\frac{wx(l - x)^2}{2}$; and the area of the parabola is $-\frac{w(l - x)^3}{6}$. In Fig. 126 the parabola is drawn with inclined base to show that these three figures make up the entire moment diagram. The moment arms of these diagrams with respect to the section

at a distance x from the left end are $\frac{l-x}{2}$, $\frac{2(l-x)}{3}$, and $\frac{3(l-x)}{4}$. The deflection at x distance from the end is given by

$$EIy = -\frac{wx^2(l-x)}{2} \times \frac{l-x}{2} - \frac{wx(l-x)^2}{2} \times \frac{2(l-x)}{3} - \frac{w(l-x)^3}{6} \times \frac{3(l-x)}{4}; \quad (4)$$

$$EIy = -\frac{w}{24}(3l^2 + 2lx + x^2)(l-x)^2;$$

$$y = -\frac{w}{24EI}(3l^4 - 4l^3x + x^4).$$

92. Stiffness of Beams.—The stiffness of a beam is the reciprocal of the deflection. The stiffness of a beam may be defined as the load which will produce unit deflection. It is not customary to express stiffness in this way; it is generally used as a relative term.

In the expression for the maximum deflection of all the beams which have been considered, the terms E and I occur in the denominator. The stiffness of a beam varies directly as the modulus of elasticity and directly as the moment of inertia of its cross-section. The moment of inertia of a rectangular section varies as the cube of the depth, consequently the stiffness of a rectangular section varies in the same ratio. All the expressions for the maximum deflection contained the cube of the length in the numerator. The stiffness of beams of the same cross-section varies inversely as the cube of their length.

Problems

1. How does the stiffness of a 4-inch by 6-inch beam compare with that of a 4-inch by 4-inch beam of the same material?
2. How does the stiffness of a 4-inch by 6-inch beam with the 6-inch side vertical compare with that of the same beam with the 4-inch side vertical?
3. How does the stiffness of a 2-inch by 12-inch beam 15 feet long compare with that of a 2-inch by 8-inch beam 10 feet long? Which is the stronger?

Miscellaneous Problems

1. A cantilever of length l carries a load W which increases uniformly from the fixed end to the free end. Find the moment at the fixed end and the deflection and slope at the free end.

   Ans. $M = -u\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$; maximum $M = -\frac{ul^3}{3} = -\frac{2Wl}{3}$; slope at end $= \frac{Wl^2}{4EI}$; $y_{max} = -\frac{11Wl^3}{60EI}$.

   Check with Article 90.

2. A beam of length l is supported at the ends and carries a load W which increases uniformly from the left end to the right end. Find the maximum moment, the equation of the elastic line, and the maximum deflection.

Ans. $R_1 = \frac{W}{3} = \frac{ul^2}{6}$; $M = \frac{ul^2x}{6} - \frac{ux^3}{6}$; maximum moment $= \frac{2Wl\sqrt{3}}{27}$ at $x = \frac{l\sqrt{3}}{3}$; slope at left support $= -\frac{7Wl^2}{180EI}$;

$$y = -\frac{u}{360EI}(3x^5 - 10l^2x^3 + 7l^4x);$$

maximum deflection at $0.5193l$ is $-\frac{0.01304Wl^3}{EI}$.

3. A curtain dam, 10 feet high, is supported laterally by vertical I-beams, each 10 feet long, which are hinged at the top and push against a stop at the bottom. These I-beams are spaced four feet apart. Find the maximum bending moment in each beam when the dam is full.

Ans. M = 192,450 inch-pounds.

CHAPTER IX

BEAMS WITH MORE THAN TWO SUPPORTS

93. Relation of Deflection to Stress.—For beams with *two* supports, including cantilevers, the moments and fiber stresses may be computed with no reference to the deflection. Such beams are *statically determinate.* When a beam has three or more supports in the same plane, it is said to be *statically indeterminate.* The reactions cannot be determined from the ordinary resolutions and moments of mechanics, but the equation of the elastic line must be taken into account. It is necessary, therefore, in all but the simplest cases, to study the deflections before the *stresses* can be calculated. Since the unit stress is the most important factor from an engineering standpoint, the reason for the prominence given to the deflection equations is evident.

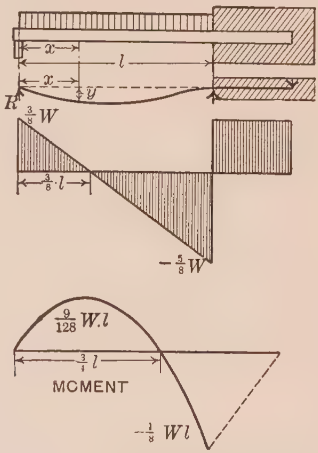


FIG. 127.—Beam fixed at one end and supported at the other.

94. Beam Fixed at One End and Supported at the Other.—Figure 127 represents a beam which is fixed at the right end and supported at the left end. The tangent at the bottom of the beam at the right end passes through the support at the left end. The load is uniformly distributed. The unknown reaction at the left support is represented by R. The moment at a distance x from the left support, which is taken as the origin of coördinates, is

$$M = Rx - \frac{wx^2}{2}. \tag{1}$$

The integration may be carried through as if R were known, and its value may be determined later from the conditions of the problem.

$$EI\frac{d^2y}{dx^2} = Rx - \frac{wx^2}{2}; \tag{2}$$

$$EI\frac{dy}{dx} = \frac{Rx^2}{2} - \frac{wx^3}{6} + \left[C_1 = -\frac{Rl^2}{2} + \frac{wl^3}{6}\right]; \tag{3}$$

$$EIy = \frac{Rx^3}{6} - \frac{wx^4}{24} - \frac{Rl^2x}{2} + \frac{wl^3x}{6} + \left[C_2 = 0\right]. \tag{4}$$

The integrations constants have been evaluated from the conditions that $\frac{dy}{dx} = 0$ when $x = l$, and $y = 0$ when $x = 0$. There remains the condition that $y = 0$ when $x = l$, which may be used to determine the unknown reaction R.

$$R = \frac{3wl}{8} = \frac{3W}{8}. \tag{5}$$

$$y = -\frac{w}{48EI}(2x^4 - 3lx^3 + l^3x), \tag{6}$$

which is the equation of the elastic line. The moment equation

$$M = \frac{3wlx}{8} - \frac{wx^2}{2}. \tag{7}$$

The reaction and moment may be found more briefly by means of the results of Chapter VIII. If the left support were removed, the beam would become a cantilever, and the deflection of the left end *downward* would be $\frac{Wl^3}{8EI}$. The reaction R, regarded as a load at the end of a cantilever, must be sufficient to deflect the beam *upward* an equal amount.

$$\frac{Rl^3}{3EI} = \frac{Wl^3}{8EI}; \; R = \frac{3W}{8}. \tag{8}$$

Problems

1. Draw shear and moment diagrams for beam fixed at one end and supported at the other. Find the moment at each dangerous section from the shear diagram and compare with the result from the equation of moments.

   Ans. Moment at dangerous sections, $\frac{9Wl}{128}$, $-\frac{Wl}{8}$.

2. How does the greatest moment, numerically, compare with that of a beam supported at the ends?
3. Find the position of maximum deflection and the value of this maximum deflection.

   Ans. Point of maximum deflection is $0.4215l$ from the left support.

Differentiating Equation (6) and equating to zero,

$$8x^3 - 9lx^2 + l^3 = 0.$$

Since the beam is horizontal at the wall, $x = l$ must satisfy this cubic. Division by the corresponding factor, $x - l$, gives a quadratic. Explain the meaning of the negative root.

4. A 4-inch by 6-inch beam for which E is 1,200,000 pounds per square inch is fixed 10 feet from the left end. The load is 60 pounds per foot. Find the deflection at the free end and find the maximum unit stress.
Ans. $y_{max} = 1.5$ inches. $S = 1{,}500$ pounds per square inch.

5. If the beam of Problem 4 is supported at the left end, what is the reaction of the support? What and where is the maximum unit stress? and what and where is the maximum deflection?
Ans. Maximum stress = 375 pounds per square inch at fixed end: maximum deflection = 0.065 inch at 50.58 inches from the supported end.

6. What would be the maximum unit stress and the maximum deflection in the beam of Problem 4 if it were supported at two points 10 feet apart and carried a load of 60 pounds per foot between the supports?
Ans. $S = 375$ pounds per square inch at the middle. $y_{max} = 5/32$ inch at the middle.

7. The beam of Problem 4 is supported at the left end. The support settles 0.5 inch when the load is applied. Find the reaction and the maximum moment. *Ans.* $R = 150$ pounds.

95. Two Equal Spans, Uniformly Distributed Load.—Figure 128 represents a continuous beam of two equal spans, each of length l. If the middle support were removed, it would become a beam of length $2l$, supported at the ends. The deflection at the middle would be $\dfrac{5w(2l)^4}{384EI}$. To bring the middle of this beam up to the line of the end supports, the force at the middle must be equal to the load at the middle which would produce this deflection.

$$\frac{R_2(2l)^3}{48EI} = \frac{5w(2l)^4}{384EI}; \tag{1}$$

$$R_2 = \frac{10wl}{8}. \tag{2}$$

Since the total load is $2wl$, $R_1 + R_3 = \dfrac{6wl}{8}$. From the symmetry, it is evident that $R_1 = R_3 = \dfrac{3wl}{8}$. It is also evident that the beam is horizontal over the middle support. The left half is exactly the same as Fig. 127, and the right half is symmetrical with the left. The end reactions have already

been shown to be the same as those of a beam which is fixed at one end and supported at the other. The equation of the elastic line and the shear and moment diagrams are also the same. The maximum moment over the second support is $-\frac{wl^2}{8}$, which is numerically the same as the moment at the middle of a beam which is supported at the ends and uniformly loaded. A beam which is continuous over three supports is no stronger than a

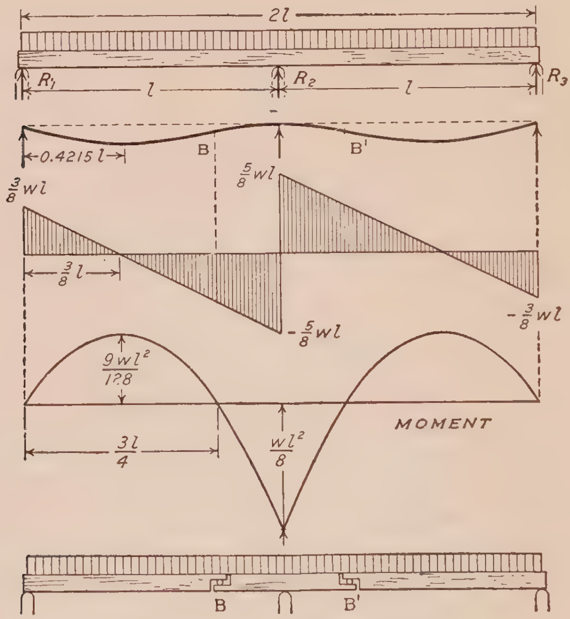


FIG. 128.—Beam with three supports.

beam of the same span which merely rests on the supports. If the beam were cut in two at the middle and each half rested on the middle support, the shear diagram would pass through zero at the middle of each span. The moment would be positive throughout the whole of each span and the moment curve would be highest at the middle of each span.

The shear diagram of Fig. 128 passes through zero at $\frac{3l}{8}$ from the ends. At $\frac{3l}{4}$ from each end the positive shear area is equal to the negative shear area and the moment is zero. The points B and B' are points of inflection. Between these points the beam

is concave downward and the moment is negative. Between the left support and B and between B' and the right support, the beam is concave upward and the moment is positive. At each point of inflection the beam might be cut in two and one portion merely rest on the other. If the beam were divided as shown at the bottom of Fig. 128, the moment, shear, and deflection at every section would be the same as in the continuous beam at the top. For instance, three-eighths of the weight of the span rests on the middle portion at B. The moment at the middle support is

$$-\frac{3wl}{8} \times \frac{l}{4} - \frac{wl}{4} \times \frac{l}{8} = -\frac{wl^2}{8}. \tag{3}$$

Wherever there is a point of inflection in a beam, the beam may be divided and the portions connected by a pin or a slight projection which will resist the shear.

Problems

1. A 6-inch by 10-inch wooden beam, 20 feet long, is supported at the ends and at the middle, and carries a load, including its own weight, of 720 pounds per foot. If the supports are all at the same level find the reaction at each and find the maximum fiber stress in the beam. Check the reactions by moments about the right support.
Ans. Maximum unit stress is 1,080 pounds per square inch over the second support.

2. If the footings for the end supports of Problem 1 are 1 foot square, what should be the area of the footing for the middle support?

3. What is the reaction of each support of the beam of Problem 1 if the middle support settles 1.44 inches below the line of the end supports and E is 1,200,000 pounds per square inch?
Ans. 4,200, 6,000, and 4,200 pounds.

4. In Problem 3, where are the dangerous sections? What is the maximum fiber stress at each?
Ans. 1,470 pounds per square inch at 70 inches from either end.

5. What would be the maximum unit stress if the beam of Problem 1 were cut in two at the middle and the halves rested on the middle support?

6. A beam of length l is supported at the ends and at a distance kl from the left end. The load is w per unit length and all supports are at the same level. Find the reaction of the intermediate support.
Ans. $R_2 = \dfrac{wl(1 - 2k^2 + k^3)}{8k(1 - k)^2}$.

7. The beam of Problem 1 has all three supports at the same level and the second support is 12 feet from the left end. Find the reaction of each support. *Ans.* 3,480, 9,300 and 1,620 pounds.

96. Beam Fixed at One End, Supported at the Other, Load Concentrated.—Figure 129 shows a beam which is fixed at the

right end and carries a load P at a distance a from the left end. The left end is held up to the line tangent to the beam at the right end by a support which exerts a reaction R. The deflection downward caused by the load P is equal to the deflection upward caused by the reaction R. These two deflections may be ob-

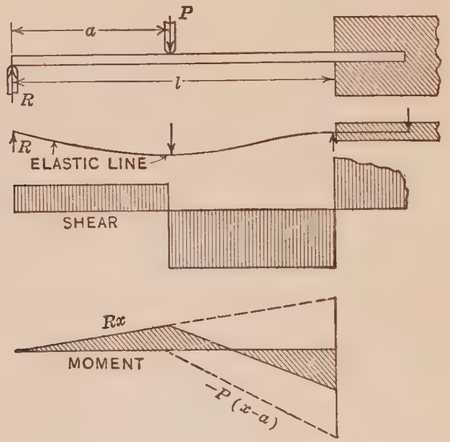


FIG. 129.—Beam fixed at one end and supported at other, load concentrated.

tained from Formula XVI and Equation (3) of Article 74, or they may be calculated directly by area moments.

$$\frac{Rl^2}{2} \times \frac{2l}{3} = \frac{P(l-a)^2}{2}\left[a + \frac{2(l-a)}{3}\right]; \tag{1}$$

$$\frac{Rl^3}{3} = \frac{P(l-a)^2}{6}(2l+a) = \frac{P}{6}(2l^3 - 3l^2a + a^3); \tag{2}$$

$$R = \frac{P(l-a)^2(2l+a)}{2l^3} = \frac{P}{2l^3}(2l^3 - 3l^2a + a^3); \tag{3}$$

$$R = \frac{P}{2}(2 - 3k + k^3), \tag{4}$$

in which $k = \frac{a}{l}$.

The equation of the elastic line may be found by adding the upward deflection caused by the reaction at the end to the downward deflection caused by the load P.

Problems

1. Find the reaction at the support, the moment under the load, and the moment at the fixed end when $k = ½$.

Ans. $R = \frac{5P}{16}$; $M = \frac{5Pl}{32}$; $M = -\frac{3Pl}{16}$.

2. A beam is fixed at one end, supported at the other, and loaded at the middle. Find the deflection under the load.

Ans. $y = -\frac{7Pl^3}{768EI}$.

3. A 6-inch by 8-inch beam, 20 feet long, is supported at the ends and at the middle and carries a load of 3,000 pounds 6 feet from the left end and an equal load 6 feet from the right end. Find the reaction of each support, if all the supports are at the same level.

Ans. 624, 4,752, and 624 pounds.

4. Solve Problem 3 if the middle support is ½ inch below the line of the end supports and the modulus of elasticity is 1,440,000 pounds per square inch. *Ans.* $R_2 = 4{,}752 - 640 = 4{,}112$ pounds.

5. A beam of length $2l$ is supported at the ends and at the middle, and carries equal loads at equal distances from each end. When the three supports are at the same level, how far should each load be placed from the end in order that the reaction at each end may be one half the reaction at the middle. Solve the cubic by trial and error. *Ans.* $k = 0.3473$.

97. Beam Fixed at Both Ends, Uniformly Loaded.—Figure 130 shows a beam which is fixed at both ends and uniformly loaded. The general moment equation is

$$M = M_0 + V_0x - \frac{wx^2}{2}, \tag{1}$$

in which M_0 and V_0 are constants at present unknown.

$$EI\frac{d^2y}{dx^2} = M_0 + V_0x - \frac{wx^2}{2}; \tag{2}$$

$$EI\frac{dy}{dx} = M_0x + \frac{V_0x^2}{2} - \frac{wx^3}{6} + [C_1 = 0]$$

From the condition that $\frac{dy}{dx} = 0$, when $x = \frac{l}{2}$,

$$\frac{M_0l}{2} + \frac{V_0l^2}{8} - \frac{wl^3}{48} = 0 \tag{3}$$

From the condition that $\frac{dy}{dx} = 0$, when $x = l$,

$$M_0l + \frac{V_0l^2}{2} - \frac{wl^3}{6} = 0 \tag{4}$$

From Equations (3) and (4), $M_0 = -\frac{wl^2}{12}, V_0 = \frac{wl}{2}$.

From the second integration,

$$EIy = -\frac{wl^2x^2}{24} + \frac{wlx^3}{12} - \frac{wx^4}{24} + [C_2 = 0]; \tag{5}$$

$$y = -\frac{wx^2}{24EI}(l - x)^2. \tag{6}$$

$$y_{\max} = -\frac{wl^4}{384EI} = -\frac{Wl^3}{384EI}. \tag{7}$$

In reinforced concrete construction, the beams are fixed to the columns, and are continuous over the intermediate supports. If the columns were perfectly rigid, the maximum moment would be $\frac{wl^2}{12}$. If the beams were not continuous and the end connections were perfectly free to turn, the maximum moment would be $\frac{wl^2}{8}$. It is customary to use an intermediate value and to assume that the maximum moment is $\frac{wl^2}{10}$.

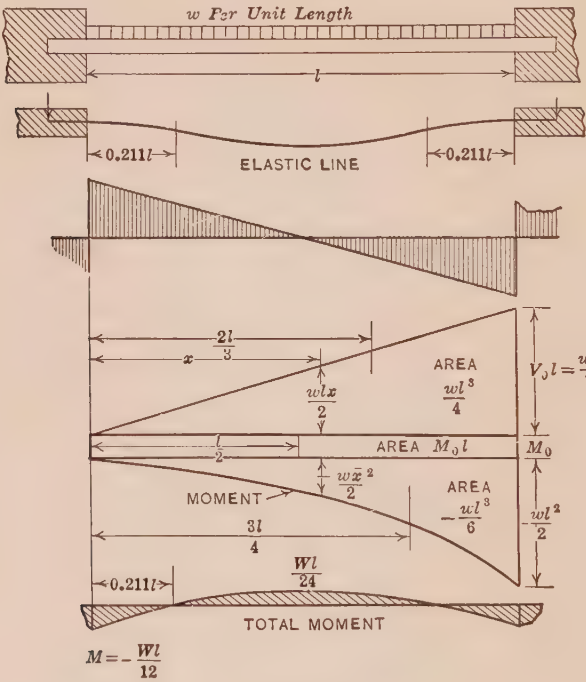


FIG. 130.

Problems

1. Show that the moment at the middle is one-half the moment at the ends.
2. Find the points of inflection. *Ans.* $0.211l$ and $0.789l$.
3. Show that the deflection at the middle is one-fifth as great as that of a beam supported at the ends.
4. Find the deflection of the left end from the tangent at the right end by area moments. Then find the deflection of the right end from the tangent at the left end, and solve for M_0 and V_0. Find the deflection of the middle from the tangent at the left end, and find the equation of the elastic line.

98. Beam Fixed at Both Ends, Load Concentrated at Any Point.—Figure 131 shows a beam which is fixed at both ends and carries a load P at a distance a ($a = kl$) from the left end. The shear and moment diagrams and the elastic line are shown in

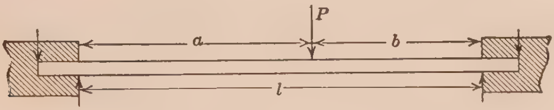


FIG. 131.—Beam fixed at ends with concentrated load.

Fig. 132. From the left end to the load, the moment is $M_0 + V_0x$. From the load to the right end, the moment is $M_0 + V_0x - P(x - a)$. These three terms are represented by the rectangle and the two triangles of Fig. 132.

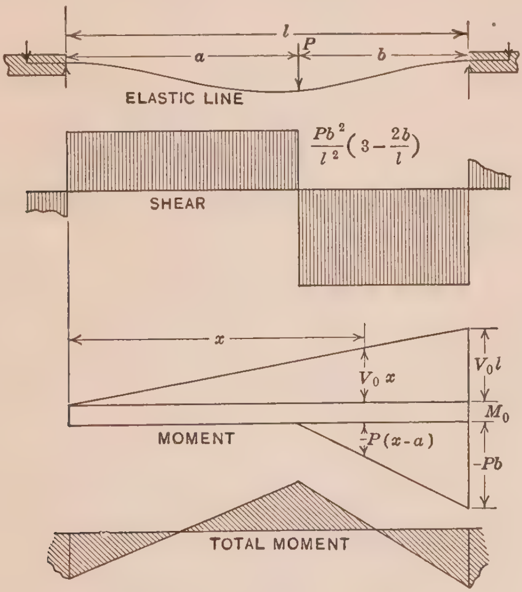


FIG. 132.—Beam fixed at ends.

To find the deflection of the left end from the tangent at the right end, Fig. 132,

$$EIy = 0 = M_0l \times \frac{l}{2} + \frac{V_0l^2}{2} \times \frac{2l}{3} - \frac{Pb^2}{2}\left(l - \frac{b}{3}\right), \qquad (1)$$

$$3M_0l^2 + 2V_0l^3 - Pb^2(3l - b) = 0. \qquad (2)$$

To find the deflection of the right end from the tangent at the left end,

$$EIy = 0 = M_0 l \times \frac{l}{2} + \frac{V_0 l^2}{2} \times \frac{l}{3} - \frac{Pb^2}{2} \times \frac{b}{3}, \tag{3}$$

$$3M_0 l^2 + V_0 l^3 - Pb^3 = 0. \tag{4}$$

From Equations (2) and (4),

$$V_0 = \frac{Pb^2}{l^2}\left(3 - \frac{2b}{l}\right) = P(1-k)^2(1+2k), \tag{5}$$

$$M_0 = -\frac{Pb^2}{l^2}(l-b) = -\frac{Pb^2 a}{l^2} = -Pk(1-k)^2 l. \tag{6}$$

The deflection at a distance x from the left end is given by

$$EIy = M_0 x \times \frac{x}{2} + \frac{V_0 x^2}{2} \times \frac{x}{3} = \frac{M_0 x^2}{2} + \frac{V_0 x^3}{6}, \tag{7}$$

provided x is not greater than a. Beyond the load, the term $\frac{P(x-a)^2}{2} \times \frac{x-a}{3}$ is subtracted and

$$EIy = \frac{M_0 x^2}{2} + \frac{V_0 x^3}{6} - \frac{P(x-a)^3}{6}. \tag{8}$$

Problems

1. If $a = \frac{l}{2}$, find the moment at the wall and under the load and find the shear at the wall.

Ans. $M = -\frac{Pl}{8}$ at the wall; $M = \frac{Pl}{8}$ at the middle.

2. In Problem 1 what is the deflection at the middle?

Ans. $y_{max} = -\frac{Pl^3}{192EI}$.

3. How does the deflection and maximum stress in a beam fixed at the ends and loaded at the middle compare with those of a beam supported at the ends and loaded at the middle?

99. Theorem of Three Moments.—The methods of the preceding articles may be applied to any number of spans or to any number of concentrated loads. However, when it is necessary to write more than two moment equations and solve for the corresponding constants, the work becomes laborious. When, as is usually the case, it is desired to find the moments, reactions, and shears, without getting the deflections, the *theorem of three moments* is of great use.

The theorem of three moments is an *algebraic equation* which expresses the relation of the moments at three successive supports of a continuous beam in terms of the length of the interven-

ing spans and the loads which they carry. In Fig. 133, the moments over the supports are represented by M_a, M_b, M_c. The length of the span from support A to support B is l_1, and from B to C it is l_2. Figure 133 represents a uniformly distributed load of w_1 pounds per unit length for the first span and w_2 pounds per unit length for the second span. The subscripts a, b, c, represent the *order* from left to right and may be applied to *any three points* in *succession.* The same is true of the subscripts 1 and 2 applied to the spans and the unit loads.

The shear adjacent to B on the side toward C is designated by V_{bc}; on the side toward A by V_{ba}.

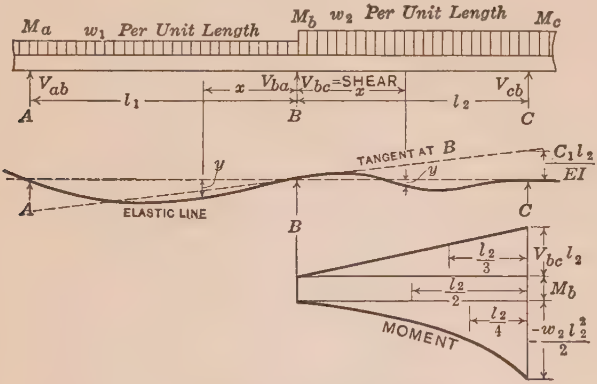


FIG. 133.—Continuous beam.

100. Theorem of Three Moments for Distributed Loads.—With the origin at the second support in Fig. 133, the differential equation for the second span is

$$EI\frac{d^2y}{dx^2} = M_b + V_{bc}x - \frac{w_2x^2}{2}. \tag{1}$$

$$EI\frac{dy}{dx} = M_bx + \frac{V_{bc}x^2}{2} - \frac{w_2x^3}{6} + C_1. \tag{2}$$

Since $\frac{C_1}{EI}$ is the value of $\frac{dy}{dx}$ when $x = 0$, it is the slope of the tangent to the beam at the middle support.

$$EIy = \frac{M_bx^2}{2} + \frac{V_{bc}x^3}{6} - \frac{w_2x^4}{24} + C_1x + [C_2 = 0]. \tag{3}$$

When $x = l_2$, $y = 0$, when these values are substituted in Equation (3), and the result divided by l_2,

$$\frac{M_bl_2}{2} + \frac{V_{bc}l_2^2}{6} - \frac{w_2l_2^3}{24} + C_1 = 0. \tag{4}$$

Equation (4) expresses the relation of the moment, shear, and slope at the left end of the second span in terms of length of the span and the load which it supports. It is desirable to replace the shear at the left of the span by the moment at the right end. From the general moment equation,

$$M_c = M_b + V_{bc}l_2 - \frac{w_2 l_2^2}{2} \tag{5}$$

When V_{bc} is eliminated from Equations (4) and (5),

$$2M_b l_2 + M_c l_2 + \frac{w_2 l_2^3}{4} + 6C_1 = 0. \tag{6}$$

The differential equation for the first span may now be written with the origin at B and with x *positive from right to left.* Since the solution is exactly the same as that for the second span, it is not necessary to go through the work. The final equation, which may be written from Equation (6), is

$$2M_b l_1 + M_a l_1 + \frac{w_1 l_1^3}{4} + 6C_3 = 0. \tag{7}$$

The slope of the tangent at B, going from right to left, is $\frac{C_3}{EI}$, therefore $C_3 = -C_1$. When Equations (6) and (7) are added, these constants are eliminated and

$$M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 = -\frac{1}{4}(w_1 l_1^3 + w_2 l_2^3). \tag{8}$$

Equation (8) is called the theorem of three moments for distributed loads. When the spans are equal and the loads per unit length in two successive spans are the same, the equation of three moments becomes,

$$M_a + 4M_b + M_c = -\frac{wl^2}{2}. \quad \text{Formula XXIII.}$$

101. Calculation of Moments for Uniform Loading.—The theorem of three moments is an algebraic relation between the moments over any three successive supports of a beam of uniform section, provided these supports remain in a straight line when loaded. For a beam with three supports, one equation may be written by the theorem, and it is necessary to know two of the moments (or to have two other independent relations) in order to solve the problem. For four supports two equations are written, the first one for supports 1, 2, 3 in order as A, B, C, of the theorem, and the second for supports 2, 3, 4. For five supports three equations are written. In every case there are two more moments than there are independent equations.

Figure 134 shows a beam with four supports and three equal spans, with no overhang at the ends. The load is w per unit length. The moments over the supports are represented by M_1, M_2, M_3, and M_4. When the theorem of three moments is applied to the first and second spans, M_a is M_1, M_b is M_2, and M_c is M_3.

$$M_1 + 4M_2 + M_3 = -\frac{wl^2}{2}. \tag{1}$$

When the theorem of three moments is applied to the second and third spans, M_a is M_2, M_b is M_3, and M_c is M_4

$$M_2 + 4M_3 + M_4 = -\frac{wl^2}{2}. \tag{2}$$

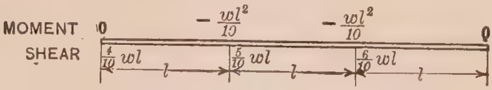


FIG. 134.—Beam of three equal spans.

Since the beam does not overhang either end support, $M_1 = 0$ and $M_4 = 0$. When these values are substituted, Equations (1) and (2) give

$$M_2 = M_3 = -\frac{wl^2}{10}.$$

Problems

1. A beam over four supports forms three equal spans as in Fig. 134, but overhangs the left support four-tenths the span and the right support two-tenths the span. Find the moment over each support.
Ans. $M_1 = -0.08wl^2$; $M_2 = -0.08wl^2$; $M_3 = -0.10wl^2$; $M_4 = -0.02wl^2$.
2. Find the moment over each support for a beam of two equal spans, with equal loads on both, with no overhang at the end supports.
Ans. $M_1 = M_3 = 0$; $M_2 = -\frac{wl^2}{8}$.
3. Find the moment over the supports for four equal spans with uniform loads on each and with no overhang.
Ans. $M_1 = M_5 = 0$; $M_2 = M_4 = -\frac{3wl^2}{28}$; $M_3 = -\frac{2wl^2}{28}$.
4. A uniformly loaded beam, 20 feet long, is supported at the ends and 8 feet from the left end. Find the moment over each support.
Ans. $M_2 = -14w$.
5. A beam 28 feet long, weighing 40 pounds per foot, is supported 4 feet from the left end, 12 feet from the left end, and 4 feet from the right end. Find the moment over each support.
Ans. $M_1 = M_3 = -320$; $M_2 = -400$ foot-pounds.

6. Solve Problem 5 if a load of 200 pounds is placed on the left end of the beam.
7. A uniformly loaded beam rests on three supports so as to have two equal spans with equal overhang on each end. What must be the ratio of overhang to span if the moments at all supports are the same?
Ans. Overhang 0.408 of the length of span.

102. Calculation of Reactions by Moments.—After the moments over the supports have been computed by the theorem, the reaction at each support may generally be determined by moments about the adjacent support.

Example

In Fig. 135, which applies to Problem 1 of Article 101, find the reaction of each support.

To find the left reaction R_1, moments are taken about a section above the second support.

$$R_1 l - 1.4wl \times 0.7l = -0.08wl^2; \quad (1)$$
$$R_1 l = 0.90wl^2; \; R_1 = 0.90wl \quad (2)$$

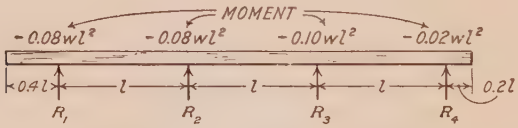


Fig. 135.—Continuous beam which overhangs supports.

To find R_2, moments are taken about a section over the third support.

$$0.90wl \times 2l + R_2 l - 2.4wl \times 1.2l = -0.10wl^2; \quad (3)$$
$$R_2 l = 0.98wl^2; \; R_2 = 0.98wl. \quad (4)$$

A similar moment equation about a section over the fourth support gives $R_3 = 1.10wl$. To find R_4, the portion of the beam to the right of the third support is used as the free body. $R_4 = 0.62wl$.

Problems

1. In the example above, check R_4 and R_3 by moments about the section over the second support. Check all reactions by vertical resolution. Check all by moments about the right end.
2. Find the reactions of the beam of Fig. 134.
Ans. $R_1 = R_4 = 0.4wl$; $R_2 = R_3 = 1.1wl$.
3. Find the reactions for Problem 2 of Article 101. Compare with Article 95.
4. Find the reactions of each support for Problem 3 of Article 101.
Ans. $R_2 = \frac{5}{4}wl$.
5. Find the reactions for Problem 4 of Article 101.
6. Find the reactions for Problem 7 of Article 101.
Ans. $0.908wl$, wl, $0.908wl$.

103. Calculation of Reactions by Means of Total Vertical Shear.—When there are more than four spans, the method of calculating the reactions which is given in Article 102 is laborious. When the ends of the beam are fixed, this solution is not practicable. A more general method will now be given, which depends upon the difference of the shear on the right and left side of the support.

If support A of Fig. 133 is taken as the origin of coördinates, and the general moment equation is applied to find the moment at B,

$$M_b = M_a + V_{ab}l_1 - \frac{w_1l_1^2}{2}, \tag{1}$$

$$V_{ab} = \frac{M_b - M_a}{l_1} + \frac{w_1l_1}{2}. \tag{2}$$

V_{ab} is the shear just to the right of *any* support;

M_a is the moment at that support, and M_b at the next one; w_1l_1 is the total uniformly distributed load between these supports. For the beam of Fig. 134, with four supports, three equal spans, and no overhang, the shear at the right of the left support is

$$V_{12} = \frac{-\dfrac{wl^2}{10} - 0}{l} + \frac{wl}{2} = 0.4wl = 0.4W.$$

At the right of the second support,

$$V_{23} = \frac{-\dfrac{wl^2}{10} + \dfrac{wl^2}{10}}{l} + \frac{wl}{2} = 0.5W.$$

In the same way, $V_{34} = 0.6\,wl = 0.6W$.

Figure 134 gives the moment over each support and the shear to the right of each support.

The shear at the left of each support is found from the definition of total vertical shear. Since the shear at the right of the first support of Fig. 134 is $0.4wl$, the shear at the left of the second support is

$$V_{21} = V_{12} - wl = 0.4wl - wl = -0.6wl.$$

The reaction of the second support in Fig. 134 is the shear at the right minus the shear at the left of the support.

$$R_2 = V_{23} - V_{21} = 0.5wl - (-0.6wl) = 1.1wl$$

Problems

1. Calculate the shear to the right of each support for Problem 1 of Article 101, and then calculate the reactions.

   Ans. $V_{12} = 0.5wl;\ V_{23} = 0.48wl;\ V_{34} = 0.58wl.$

2. Find the shear at the right of each support for a uniformly loaded beam of four equal spans with no overhang. Then find the shear to the left of each support and find each reaction. Compare with Fig. 136.

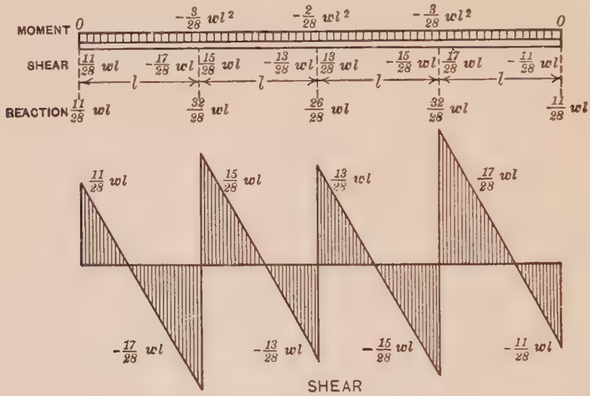


FIG. 136.—Beam of four spans.

3. Find the moments, shears, and reactions for a beam of five equal spans, uniformly loaded with no overhang. Draw diagrams similar to Fig. 136.
4. From Fig. 136, find the maximum moment and the location of the point of inflection in the first span.
5. A beam carrying a uniformly distributed load rests on three supports spaced 10 feet apart. How much should it overhang the outer supports in order that the reactions at all the supports shall be the same?

   Ans. 4.4 feet.

104. Theorem of Three Moments for Concentrated Loads.—Figure 137 shows a continuous beam, which carries a load P on the left span at a distance a from the second support, and a load Q on the right span at a distance c from the second support. The broken straight line at an angle θ with the horizontal is tangent to the elastic line over the second support. (On the figure, this tangent line is above the horizontal at the right of the second support and below the horizontal at the left of that support. It would make no difference in the final result if the slope were reversed.)

The deflection of this tangent line *downward* from the left support is θl_1. The deflection of the elastic line at the left support *upward* from the straight line which is tangent over the

second support is the moment, divided by EI, of the complete moment diagram for the first span. In Fig. 137, the moment diagram for the left span has been drawn with the origin of coördinates at the left support. The general moment equation which corres-

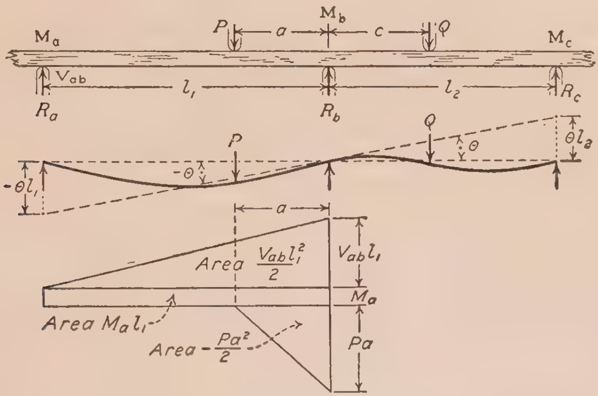


FIG. 137.—Continuous beam with concentrated loads.

ponds with this diagram has been expressed in terms of M_a, the moment at the left support; V_{ab}, the shear at the right of the left support; and the load P at a distance a from the second support. The moment of the diagram with respect to the left support is

$$M_a l_1 \times \frac{l_1}{2} + \frac{V_{ab}l^2}{2} \times \frac{2l_1}{3} - \frac{Pa^2}{2}\left(l_1 - \frac{a}{3}\right); \tag{1}$$

Since the total deflection at the left support is equal to the deflection downward of the tangent at the second support,

$$EIy = 0 = -EI\theta l_1 + \frac{M_a l_1^2}{2} + \frac{V_{ab}l_1^3}{3} - \frac{Pa^2 l_1}{2} + \frac{Pa^3}{6}. \tag{2}$$

When Equation (2) is multiplied by 6 and divided by l_1, the result is

$$-6EI\theta + 3M_a l_1 + 2V_{ab}l_1^2 - 3Pa^2 + \frac{Pa^3}{l_1} = 0. \tag{3}$$

From the general moment equation,

$$M_b = M_a + V_{ab}l_1 - Pa;$$
$$V_{ab}l_1 = M_b - M_a + Pa. \tag{4}$$

When this value of the shear is substituted in Equation (3) the result is

$$-6EI\theta + 3M_al_1 + 2(M_b - M_a)l_1 + 2Pal_1 - 3Pa^2 + \frac{Pa^3}{l_1} = 0; \quad (5)$$

$$-6EI\theta + M_al_1 + 2M_bl_1 = -2Pal_1 + 3Pa^2 - \frac{Pa^3}{l_1}. \quad (6)$$

For the second span, the deflection of the tangent line, θl_2, is positive. From the analogy with Equation (6), the equation for this span may be written

$$6EI\theta + M_cl_2 + 2M_bl_2 = -2Qcl_2 + 3Qc^2 - \frac{Qc^3}{l_2}. \quad (7)$$

When Equations (6) and (7) are added, θ is eliminated and

$$M_al_1 + 2M_b(l_1 + l_2) + M_cl_2 = -Pa\left(2l_1 - 3a + \frac{a^2}{l_1}\right) - Qc\left(2l_2 - 3c + \frac{c^2}{l_2}\right). \quad (8)$$

If $\frac{a}{l_1} = k_1$, and $\frac{c}{l_2} = k_2$, Equation (8) becomes

$$M_al_1 + 2M_b(l_1 + l_2) + M_cl_2 = -Pk_1l_1^2(2 - 3k_1 + k_1^2) - Qk_2l_2^2(2 - 3k_2 + k_2^2). \quad (9)$$

Equation (8) is the theorem of three moments for concentrated loads. The terms to the left of the equality sign are identical with those of the equation for distributed loads. When distributed and concentrated loads are combined, the equation of three moments is obtained by adding the second member of Equation (8) of Article 100 to the second member of Equation (8) or (9) of Article 104.

When there are several concentrated loads on each span, the theorem of three moments is

$$M_al_1 + 2M_b(l_1 + l_2) + M_cl_2 = -\Sigma Pk_1l_1^2(2 - 3k_1 + k_1^2) - \Sigma Qk_2l_2^2(2 - 3k_2 + k_2^2), \quad (10)$$

in which $Pk_1l^2{}_1(2 - 3k_1 + k^2{}_1)$ is the summation of a series of terms of the form of $l_1^2[P_1(2k_1 - 3k_1^2 + k_1^3) + P_3(2k_3 - 3k_3^2 + k_3^3) + \text{etc.}]$ In this expression, P_1, P_3 etc. are loads on the first span at distances k_1l_1, k_2l_2 respectively from the middle support.

Example

A uniform beam, 37 feet long, is supported at the ends, at 10 feet from the left end, and at 12 feet from the right end. It carries a load of 1,000 pounds 7 feet from the left end, a load of 800 pounds 16 feet from the left end, a load

of 540 pounds 8 feet from the right end, and a load of 800 pounds 6 feet from the right end. Find the moment over each support and the reaction of each support.

For the first and second spans, $k_1 = 0.3$, $k_2 = 0.4$, $P = 1{,}000$, $Q = 800$, $l_1 = 10$ feet, $l_2 = 15$ feet.

$$0 + 50M_2 + 15M_3 = -1{,}000 \times 100 \times 0.3(2 - 0.9 + 0.09) - 800 \times 225 \times 0.4(2 - 1.2 + 0.16), \quad 50M_2 + 15M_3 = -104{,}820.$$

From the second and third spans

$$15M_2 + 54M_3 = -800 \times 225 \times 0.6(2 - 1.8 + 0.36) - 540 \times 144 \times \frac{1}{3}\left(2 - 1 + \frac{1}{9}\right) - 800 \times 144 \times \frac{1}{2}\left(2 - \frac{3}{2} + \frac{1}{4}\right).$$

$$15M_2 + 54M_3 = -132{,}480.$$

From these equations, $M_2 = -1{,}484$, $M_3 = -2{,}041$ foot-pounds.

The shear at the right of each support (except the fourth) may be found by the general moment equation and the reactions may be calculated from the difference of shear on opposite sides of each support. The reactions may also be calculated by moments about each support. At the second support,

$$10R_1 - 3{,}000 = -1{,}484,$$
$$R_1 = 151.6 \text{ pounds}$$

Problems

1. Find the reaction at each support for the example above.

Ans. 152, 1,291, 1,287, and 410 pounds.

2. A beam 24 feet long is supported at the ends and 10 feet from the left end. It carries a load of 480 pounds 4 feet from the left end and a load of 336 pounds 7 feet from the right end. Find the moment over the middle support and the reaction of each support.

Ans. $M_2 = -850.5$ foot-pounds; $R_1 = 202.95$ pounds; $R_2 = 505.8$ pounds; $R_3 = 107.25$ pounds.

3. A shaft 28 feet long, which weighs 10 pounds per foot, is supported 2 feet from the left end, 12 feet from the left end, and 2 feet from the right end. It carries a load of 120 pounds 1 foot from the left end, 480 pounds 8 feet from the left end, 280 pounds 6 feet from the right end, and 240 pounds 1 foot from the right end. Find the reaction of each support.

4. A beam of length $2l$ is supported at the ends and at the middle and carries a load P at a distance $\frac{l}{3}$ from the left end and an equal load at the same distance from the right end. Find the moment over the middle support and the reaction of each support.

Ans. $M_2 = -\frac{4Pl}{27}$; $R_1 = R_3 = \frac{14P}{27}$; $R_2 = \frac{26P}{27}$.

5. Solve Problem 4 without the use of the theorem of three moments. Imagine the middle support removed and find the deflection at the

middle which is caused by the two symmetrically placed loads. Then find the load at the middle of a beam supported at the ends which will produce this same deflection.

105. Deflection When Bending Moment is Not Parallel to Principal Axis of Inertia.—When the bending moment was not parallel to one of the principal axes of inertia it was found necessary to resolve the moment or the forces parallel to these axes before calculating the fiber stress.

In the same way, to find the deflection, the forces must be resolved into components and the deflections calculated parallel to each of these two axes. The resultant deflection at any point is the vector sum of the components.

Example

A 2-inch by 3-inch wooden cantilever, 10 feet long, has the 3-inch faces at an angle of 35 degrees with the horizontal. Find the magnitude and direction of the deflection at the end due to a load of 20 pounds on the end, if E is 1,200,000 pounds per square inch.

The components of the load are 20 cos 35 degrees and 20 sin 35 degrees. The corresponding moments of inertia are 2 inches^4 and 4.5 inches^4, respectively. The deflection perpendicular to the 3-inch faces is $4.8 \times 0.8192 = 3.932$ inches.

The deflection parallel to the 3-inch faces is

$$\frac{32}{15} \times 0.5736 = 1.224 \text{ inches.}$$

The angle ϕ which the resultant deflection makes with the 2-inch faces is given by

$$\tan \phi = \frac{1.224}{3.932} = \frac{32 \sin 35°}{15 \times 4.8 \cos 35°} = \frac{2}{4.5} \tan 35° = 0.3112.$$

$$\phi = 17° \ 17'.$$

Resultant deflection $= 3.932 \sec \phi = 4.118$ inches, at 17 degrees 43 minutes with the vertical.

Problems

1. A 3-inch by 4-inch wooden cantilever 5 feet long, is placed with one diagonal horizontal. Find the deflection at the end due to a load of 90 pounds on the end, if E is 1,500,000 pounds per square inch.
2. Two 6-inch by 4-inch by 1-inch angles are placed with the 6-inch legs vertical so as to form parallel cantilevers 10 feet in length. If the 4-inch legs are in opposite directions and away from each other, and E is 29,000,000 pounds per square inch, how much will the ends separate when a load of 600 pounds is placed on each cantilever?

106. Deflection from Moments in More Than One Plane.—When the forces acting on a beam are not all parallel to one plane which passes through the beam, it is necessary to resolve the

forces into components parallel to two axes which are perpendicular to each other and to the length of the beam. If the beam is circular, square, or of any other section for which the moment of inertia is the same in every direction, these axes may be taken in any convenient way. For all other sections the resolutions must be made parallel to one of the principal axes of inertia. The two components of the deflection at any point are calculated separately, and the resultant deflection found from their vector sum.

Example

A 3-inch solid shaft, weighing 24 pounds per foot, is 10 feet long and is supported at the ends. A pulley weighing 160 pounds is 3 feet from the left end, and is subjected to a pull of 400 pounds 30 degrees below the horizontal in a plane perpendicular to the length of the shaft. Find the deflection at the pulley, if E is 29,000,000 pounds per square inch.

Resolving vertically, the total vertical load at the pulley is 360 pounds. The horizontal pull is 346.4 pounds. The deflections at 36 inches from one end are:

From concentrated load of 360 pounds.........	0.0793 inch.
From load of 2 pounds per inch...............	0.0381 inch.
Total vertical deflection....................	0.1174 inch.

The horizontal deflection from load of 346.4 pounds is 0.0763 inch.

Problem

1. A 10-inch, 15-pound channel 20 feet long is supported at the ends with the web inclined 20 degrees to the vertical. It carries a vertical load of 300 pounds per foot and a load of 400 pounds per foot perpendicular to the flange. Find the deflection and fiber stress at the middle.

Miscellaneous Examples

1. Solve Problem 2 of Article 102 by means of Problem 6 of Article 79 and the equation of the elastic line for a beam supported at the ends with uniformly distributed load.
2. Solve Problem 5 of Article 102, by means of the equation of the elastic line for a beam supported at the ends with a uniformly distributed load, and the equation for the deflection under the load for beam with a concentrated load at any point.
3. Write the theorem of three moments for uniformly distributed loads when the moment of inertia of the left span is twice that of the right span.

   Ans. $M_a l_1 + 2M_b(l_1 + 2l_2) + 2M_c l_2 = -\dfrac{w_1 l_1^3}{4} - \dfrac{w l_2^3}{2}.$
4. Three 12-inch, 31.5-pound I-beams, each 12 feet long, are 10 feet apart and are supported at the ends. Another 12-inch, 31.5-pound I-beam, 20 feet long is placed across the middle of the three beams. A load of 6,000 pounds is placed 5 feet from each end of the transverse beam. Find the load on each of the three beams and the maximum fiber stress in all four. *Ans.* Load on 12-foot beams, 2,395, 7,210, and 2,395 pounds.

CHAPTER X

SHEAR IN BEAMS

107. Direction of Shear.—The total vertical shear in a beam is calculated by the methods of Article 51, but this gives no information in regard to the distribution of the shearing stress in the section. In Article 30 it was shown that shearing stresses occur in pairs, and that a small block subjected to shearing stress of given intensity along two parallel faces is subjected to a shearing stress of the same intensity along two other faces at right angles to these.

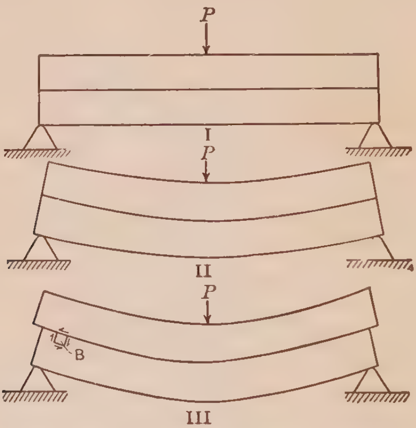


FIG. 138.—Horizontal shear in beams.

Figure 138, I, represents a beam made by placing one plank on top of another. Figure 138, II, is the same beam under load, provided that the planks are held from slipping with reference to each other by being glued or bolted together to form a single beam. If the planks are free to move, they take the form III, in which the upper plank is moved outward over the lower one at each end. A small block B in the upper portion of the lower plank may be treated as a free body. The plank above this block has been displaced toward the left. If the planks were glued together, the upper plank would have exerted a horizontal

shearing stress upon the upper surface of the block. To prevent rotation there must be a vertical shear upward at the left side. The actual shearing stresses upon this block from the surrounding material, if the upper plank were glued to the lower, would take the directions of the arrows.

The shear at the left of the block is vertically upward, which is the direction of the external shear. If a block were taken to the right of the load P, it would be found that the shear on its left side is vertically downward, which is the direction of the vertical shear in that part of the beam. One of the planks of Fig. 138 may be thicker than the other, but the *direction* of the shear will remain the same.

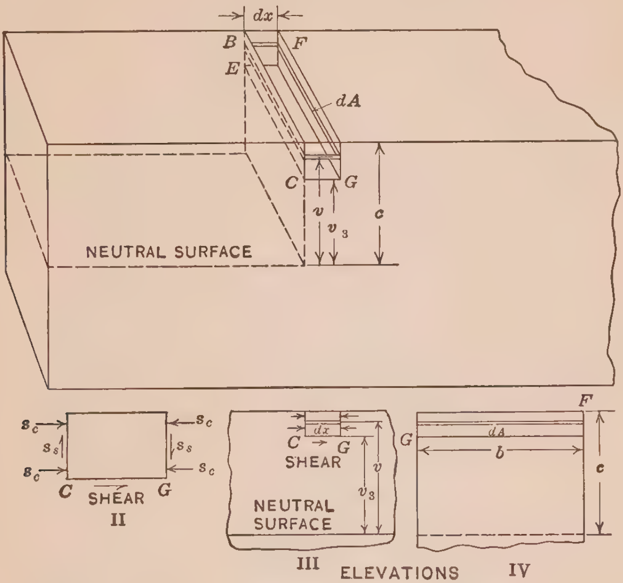


FIG. 139.—Horizontal shear in rectangular section.

108. Intensity of Shearing Stress.—Figure 139 represents a part of a beam subjected to vertical shear and to a bending moment. The shear is assumed to be positive from left to right and the moment is assumed to produce compression in the fibers above the neutral surface. A small block is shown extending across the beam between vertical planes dx apart and reaching from the top of the beam to a horizontal plane at a distance v_3 from the neutral surface. Two elevations of this block and the adjoining parts of the beam are shown in Fig. 139, III and IV, and an en-

larged elevation of the block in Fig. 139, II. The block is in equilibrium under the action of the compressive stress on the ends (the rectangles whose diagonals are CB and GF), the vertical shearing stress on the same surfaces, the horizontal shear from the material below (on the rectangle GE), and vertical compression or tension across the base.

In Fig. 139, a filament of cross-section dA and length dx extends through the block parallel to the neutral surface. The unit compressive stress on the left end of this filament is $\frac{M_1 v}{I_1}$, in which M_1 is the bending moment at the section, and I_1 is the moment of inertia with respect to the neutral axis of the entire cross-section of the beam. The compression of the left end of the filament is $\frac{M_1 v}{I_1}\,dA$. The total compression on the left end of the block is the integral from v_3 to c of the compression on the left end of the filament.

$$\text{Total compression on left end} = \frac{M_1}{I_1}\int_{v_3}^{c} v\,dA. \tag{1}$$

$$\text{Total compression on right end} = \frac{M_2}{I_2}\int_{v_3}^{c} v\,dA. \tag{2}$$

The resultant horizontal push on the block in the direction of the length of the beam is the difference of these integrals (1) and (2). If the section of the beam is uniform, $I_1 = I_2$, and v_3 and c, are the same for both expressions. The resultant horizontal push on the block is

$$\frac{M_2 - M_1}{I}\int_{v_3}^{c} v\,dA \tag{3}$$

This resultant horizontal force must be balanced by the horizontal shear at the bottom of the block. If the breadth CE at the bottom of the block is b, the total area in horizontal shear is $b\,dx$, and the total shear is $s_s b\,dx$. When the resultant horizontal compression is equated with the horizontal shear at the bottom of the block,

$$s_s b\,dx = \frac{M_2 - M_1}{I}\int_{v_3}^{c} v\,dA; \tag{4}$$

$$s_s = \frac{M_2 - M_1}{Ib\,dx}\int_{v_3}^{c} v\,dA. \tag{5}$$

Since $M_2 - M_1$ is equal to dM,

$$\frac{M_2 - M_1}{dx} = \frac{dM}{dx} = V, \tag{6}$$

in which V is the total vertical shear.

$$s_s = \frac{V}{Ib}\int_{v_3}^{c} v\,dA, \tag{7}$$

in which s_s equals the unit horizontal shear at a distance v_3 from the neutral axis and also equals the unit vertical shear at the same place. The term $\int_{v_3}^{c} v\,dA$ is the moment of the area of the end of the block with respect to the neutral axis.

$$\bar{v} = \frac{\int_{v_3}^{c} v\,dA}{A}; \int_{v_3}^{c} v\,dA = \bar{v}A. \tag{8}$$

When the area and location of the center of gravity of the portion of the plane section above the line CE are known, the integral may be replaced by the equivalent expression of (8). Equation (7) then becomes

$$s_s = \frac{V}{Ib}\bar{v}A. \qquad \text{Formula XXIV.}$$

These equations have been derived for compressive stress, they apply also when the stress is tensile. They are based on the assumption that the unit stress varies as the distance from the neutral surface. They are strictly valid, therefore, only when the unit stress in the outer fibers is below the proportional elastic limit. Since the maximum shear in a beam usually occurs at sections where the bending moment is small and the unit stress is below the elastic limit, it is seldom necessary to make a correction on this account.

Formula XXIV gives the unit *horizontal* shearing stress in a beam in terms of the total vertical shear V. It has been shown in Article 30 that the unit vertical shearing stress on an infinitesimal block is equal to the unit horizontal shearing stress. The unit horizontal shearing stress in a beam increases from the outer fibers to the neutral surface; and the unit vertical shearing stress changes in the same way.

Formula XXIV gives the average unit shearing stress in the horizontal plane GE of Fig. 139. If the section of the beam is not rectangular, the unit shearing stress may not be uniform

over the horizontal surface. Figure 140, I, is a circular section, in which AD is the trace of a horizontal plane. The short lines are traces of planes in which the shear is transmitted from one side of AD to the other. At the middle of AD the shear is transmitted from a filament above the plane to a filament directly below it,

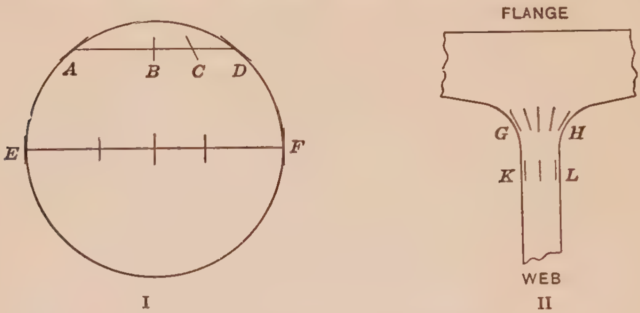


FIG. 140.—Shear in curved sections.

and the line B is vertical. At A and D, the shear is transmitted from a filament on the outer surface above the plane to a slightly larger filament, also on the outer surface, below the plane. The short lines at A and D are tangent to the section. At the diameter EF, the shear is transmitted from a filament above the plane to an equal filament directly below it, and it is customary to assume that the distribution is uniform.

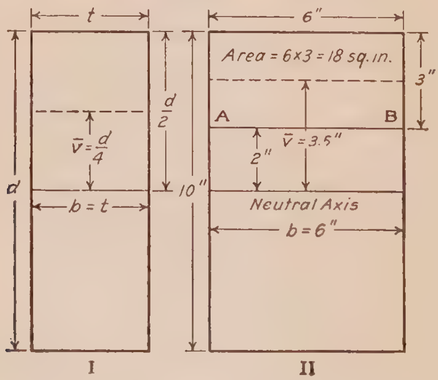


FIG. 141.—Rectangular section in shear.

Figure 140, II, is part of an I-beam section. At the plane where the web joins the flange, there must be a great difference in the intensity of the shearing stress. At KL, at some little dis-

tance down the web, the shearing stress becomes practically uniform over the section.

The unit shearing stress at the neutral surface of a rectangular section is one-half greater than the average unit vertical shear. If V is the total vertical shear, the average vertical shear in a rectangular beam of width t and height d is $\frac{V}{td}$. In Formula XXIV, $b = t$, $A = \frac{td}{2}$, $\bar{v} = \frac{d}{4}$, and $I = \frac{td^3}{12}$. (Fig. 141, I.)

$$s_s = \frac{V}{t} \times \frac{12}{td^3} \times \frac{d}{4} \times \frac{td}{2} = \frac{3V}{2\,td}, \tag{9}$$

which is three halves as great as the average unit vertical shear.

Example

Find the unit shearing stress in a 6-inch by 10-inch rectangular section, at a plane 3 inches from the top, when the total vertical shear is 6,000 pounds. (Fig. 141, II.)

$$\frac{V}{Ib} = \frac{6{,}000}{500 \times 6} = 2; \bar{v}A = 3.5 \times 18 = 63;$$

$$s_s = 2 \times 63 = 126 \text{ pounds per square inch.}$$

In this solution, the unit shearing stress in the plane AB of Fig. 141, II, has been found by means of the area above the plane. The area between AB and the bottom of the section might have been used. With that area, $A = 42$ and $\bar{v} = 3.5 - 2 = 1.5$ inches. $\bar{v}A = 1.5 \times 42 = 63$, which is the same as the preceding result.

Problems

1. Find the unit shearing stress in the above example at 1 inch above the neutral surface. *Ans.* $s_s = 2 \times 72 = 144$ pounds per square inch.
2. Find the unit shearing stress in the above example at the neutral surface. Solve by Formula XXIV and check by means of the average vertical shearing stress.
3. A 6-inch by 8-inch beam, 12 feet long, is supported at the ends and carries a load 2,880 pounds 5 feet from the left end. Find the unit shearing stress at the neutral axis of sections at which the total vertical shear is the greatest. *Ans.* $s_s = 52.5$ pounds per square inch.
4. In Problem 3, find the unit shearing stress 2 inches and 3 inches above the neutral surface.
5. In a beam of solid circular section, what is the ratio of the unit shearing stress at the neutral surface to the average unit shearing stress, assuming that the unit stress is uniform? *Ans* 4:3.
6. The American Railway Engineering Association specifications give 120 pounds per square inch as the allowable longitudinal shearing stress in

beams of long leaf yellow pine. The same specifications give 1,300 pounds per square inch as the allowable extreme fiber stress in bending. (See Cambria.) Find the total safe load at the middle of a 6-inch by 8-inch beam which is supported at the ends, for lengths of 3 feet, 5 feet, and 10 feet.

7. Solve Problem 6 if the loads are uniformly distributed.

8.* A 7-inch by 14-inch beam of long-leaf yellow pine, placed on supports 13 feet 6 inches apart, was subjected to equal loads at points 4 feet 6 inches from the supports. When the total load was 57,500 pounds, the beam failed by shear at the neutral axis at one end. Find the ultimate shearing strength of this timber parallel to the grain. Compare the result with the figures given by the United States Department of Agriculture (see handbook). *Ans.* 440 pounds per square inch.

9.* A 7-inch by 16-inch beam of Douglas fir, supported at points 13 feet 6 inches apart and loaded at the third points with equal loads, failed by shear when the total load was 45,000 pounds. Find the ultimate shearing strength of this timber parallel to the grain.

Ans. 301 pounds per square inch.

10. Timber having an allowable unit shearing stress, parallel to the grain, of 100 pounds per square inch, and an allowable bending stress of 1,000 pounds per square inch, is used for beams supported at the ends and loaded at the middle. Below what length will the shear determine the load in a 4-inch by 6-inch beam?

The total vertical shear at either end is,

$$V = 24 \times \tfrac{2}{3} \times 100 = 1{,}600 \text{ pounds.}$$

The maximum moment under the load is,

$$M = 1{,}000 \times \frac{bd^2}{6} = 24{,}000 \text{ inch-pounds}$$

$$1{,}600 \times \frac{l}{2} = 24{,}000.$$

$$\frac{l}{2} = 15 \text{ inches, } l = 30 \text{ inches.}$$

11. The timber of Problem 10 is used to support a load which is uniformly distributed. Below what length will the shear determine the load in the case of a 4-inch by 6-inch beam? Solve also for a 6-inch by 10-inch beam, and for an 8-inch by 6-inch beam.

Ans. 5 feet, 8 feet 4 inches, 5 feet.

12. In Problems 8 and 9 what was the maximum bending stress?

109. Shearing Stress in I-beams.—It is customary to calculate the unit shearing stress in the web of an I-beam by dividing the total vertical shear by the area of cross-section of the web regarded

* Problems 8 and 9 are from tests made by Prof. A. N. Talbot, described in *Bulletin* No. 41 of the Engineering Experiment Station of The University of Illinois.

as extending the entire depth of the beam. If t is the thickness of the web and d is the depth of the beam, it is assumed that

$$\text{average unit shearing stress} = \frac{\text{total vertical shear}}{td}.$$

In a 12-inch 31.5-pound I-beam, Figure 142, the thickness of the web is 0.35 inch; the area td is 4.2 square inches, and the average unit shearing stress, as computed by this method, is $0.238V$. To find the unit shearing stress at the neutral surface the upper half of the section is divided in a vertical rectangle, a horizontal rectangle, and two triangles, and the moment of each area is computed.

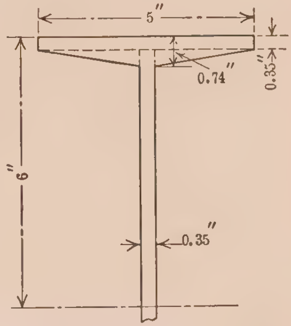


FIG. 142.—I-beam section.

	A	$\bar{v}$	$A\bar{v}$
Horizontal rectangle...	1.750	5.825	10.194
Two triangles.........	0.907	5.520	5.006
Vertical rectangle......	1.977	2.825	5.585
Total.............	.....	.....	20.785

$$\frac{V}{Ib}\bar{v}A = \frac{20.785V}{215.8 \times 0.35} = 0.275V.$$

To find the unit shearing stress in a plane 5 inches above the neutral surface, which is a little below the flange, the moment of the vertical rectangle, 5 inches high and 0.35 inch thick, is subtracted from the moment of the entire upper half of the section.

$$\bar{v}A = 20.785 - 5 \times 0.35 \times 2.5 = 20.785 - 4.375 = 16.409.$$

$$s_s = \frac{16.409V}{215.8 \times 0.35} = 0.217V.$$

The average of $0.275V$ and $0.217V$ is $0.246V$, which differs very little from $0.238V$. It is evident that the method of calculating average unit shear in an I-beam section gives a result which is practically correct.

Problem

Calculate the unit shearing stress in terms of the total shear in the web of a 10-inch 25-pound I-beam at the neutral surface and at the bottom of the flange.

Ans. $s_s = 0.368V$ at the neutral surface. $s_s = 0.291V$ at the bottom of the flange.

110. Relation of Shearing Stress to Stress Distribution Diagram.—When the vertical shear is constant from the end of a beam to some definite section, the unit shearing stress at the neutral surface may be found by dividing the total compression or tension at the section by the shear area from the section to the end of the beam. For instance, a 6-inch by 10-inch beam, 5 feet long, is supported at the ends and carries a load of 7,200 pounds at the middle. If the weight of the beam is neglected, the vertical shear from the left end to the middle is 3,600 pounds. The unit compressive stress in the top fibers at the middle is

$$S = \frac{3{,}600 \times 30}{100} = 1{,}080 \text{ pounds per square inch,}$$

The average compressive stress over the upper half of the section is 540 pounds per square inch, and the total compression is $540 \times 30 = 16{,}200$ pounds. When the portion of the beam above the neutral surface extending from the middle to the left end is taken as a free body, the only horizontal forces are this compression of 16,200 pounds and the shear at the neutral surface. The shear area is $6 \times 30 = 180$ square inches.

$$180s_s = 16{,}200; \quad s_s = 90 \text{ pounds per square inch.}$$

Instead of the section at the middle, any section to the left of the middle might be used. At 20 inches from the end, for instance, the maximum stress is 720, the total is 10,800 and the unit shearing stress is $10{,}800 \div 120 = 90$ pounds per square inch. If the beam extends beyond the supports, the shear area is increased and the unit shearing stress is diminished. For instance, if the beam projects 6 inches beyond the left support, the area from the middle to the end is 216 square inches, and the computed value of the unit shearing stress is 75 pounds per square inch. The extension of the beam beyond the support acts as an anchor, and reduces the danger of failure by shear.

If the shear is desired in a plane 2 inches above the neutral surface, the total compression must be found between this plane and the top of the beam. The unit stress at the middle section 2 inches from the neutral surface is 432 pounds. The average from this plane to the top is $\frac{432 + 1{,}080}{2} = 756$. The total compression is $756 \times 18 = 13{,}608$ pounds, and $s_s = 13{,}608 \div 180 = 75.6$ pounds per square inch.

Since the total tension or compression above a given surface is proportional to the area of the stress distribution diagram above that surface, it follows that the stress distribution diagram is a measure of the variation of shearing stress in a section. In Fig. 143, which is the stress distribution diagram for a rectangular section, the total force is measured by the area of the entire triangle. The total force above a plane which is one-fourth the depth of the section above the neutral axis is measured by the trapezoid above this plane. Since the triangle below the plane has one-fourth the area of the entire triangle, the trapezoid has three-fourths the area of the entire triangle. In a rectangular section, the unit shearing stress half way between the neutral surface and the top or bottom of the beam is three-fourths the unit shearing stress at the neutral surface.

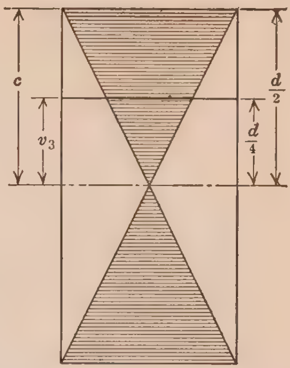


FIG. 143.—Distribution diagram.

In an I-beam most of the shaded area in the stress-distribution diagram is in the flange. The small shaded area in the web measures the difference between the shearing stress at the neutral axis and that at the bottom of the flange.

Example

A 4-inch by 10-inch rectangular beam is subjected to a total vertical shear of 2,000 pounds. Find the unit shearing stress at each inch above the neutral axis by means of the stress-distribution diagram.

The average unit shearing stress is 50 pounds per square inch, and the unit shearing stress at the neutral surface is $\frac{3}{2} \times 50 = 75$ pounds per square inch. The area of the stress-distribution triangle above the neutral axis is 10 square inches, and the area of the similar triangle below the 1-inch line is $\frac{1}{25}$ as great. The area of the diagram above the 1-inch line is $\frac{24}{25}$ of that of the total triangle. The unit shearing stress at 1 inch from the neutral axis is $\frac{24}{25} \times 75 = 72$ pounds per square inch. At 2 inches the unit stress is $\frac{4}{25} \times 75 = 12$ pounds less than at the neutral surface.

Problems

1. The unit shearing stress in a 5-inch by 12-inch beam at the neutral surface is 72 pounds per square inch. What is the unit shearing stress at each inch above or below that surface? Solve without writing.

2. An 18-inch, 55-pound I-beam, 10 feet long, is supported at the ends and carries a load at the middle which makes the maximum unit stress 16,000 pounds per square inch. The area of the stress distribution diagram above the neutral surface is 5.74 square inches. What is the total compression in the upper half of the beam. What is the horizontal shearing stress at the neutral surface if the beam does not project beyond the supports? If the area of the stress distribution diagram from the neutral axis to a plane 8 inches above this axis is 1.64 square inches, what is the unit shearing stress at this plane?

111. Failure of Beams.—The nature of the failure in a beam depends principally upon the relative ultimate strength of the material in the different directions and the value of the different

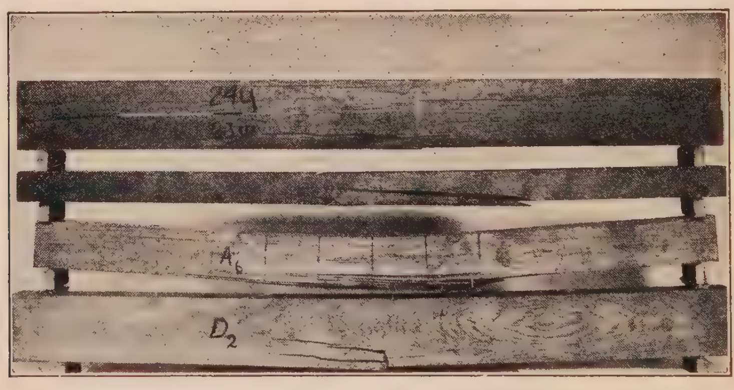


Fig. 144.—Failure of timber beams.

maximum stresses. In a beam which is short relative to its depth, the unit tensile and compressive stresses at the dangerous section are small compared with the unit shearing stress at the neutral surface at the ends. Owing to the fact that timber has a small shearing strength parallel to the grain, such a beam, if made of timber, will usually fail by shear. Figure 144 shows four wooden beams each about 40 inches long. The upper beam is a yellow pine beam glued to a white pine beam. The total depth was 3.80 inches and breadth 1.57 inches. The beam was supported at points 36 inches apart and loaded at the third points; this beam failed by longitudinal shear at one end when the total load was 1,950 pounds. The failure followed the glued surface but began in the white pine.

Beams of brittle material, such as cast iron, hard steel, stone, or concrete which is not reinforced, fail by tension. Beams of soft

steel fail by buckling on the compression side or by buckling of the web in the case of I-beam sections.

112. Deflection Which Is Due to Shear.—When a beam is bent, a part of the deflection, unless the moment is constant, is due to shear. If the unit shearing stress across a section were constant, the shearing deformation in a length dx would be $\frac{s_s}{E_s}\,dx$, and the total deflection in a length l would be

$$y_s = \frac{1}{E_s}\int_0^l s_s dx. \tag{1}$$

If s_s is constant throughout the length l, this becomes,

$$y_s = \frac{s_s l}{E_s}. \tag{2}$$

In an I-beam section the unit shearing stress is *assumed* to be constant and the equations above apply to give an *approximate* result.

Example I

Find the deflection of a 10-inch, 25-pound I-beam, supported at points 12 inches apart, and loaded with 49,600 pounds at the middle of the span. $E_s = 12{,}000{,}000$, $E = 29{,}000{,}000$.

The vertical shear is 24,800 pounds. Since the web area is 3.1 square inches, the unit shearing stress is 8,000 pounds per square inch. If the middle is regarded as fixed, the shear of either end upward is

$$y_s = \frac{8{,}000 \times 6}{12{,}000{,}000} = 0.004 \text{ inch.}$$

The deflection which is due to bending is

$$y = \frac{49{,}600 \times 12^3}{48 \times 29{,}000{,}000 \times 122.1} = 0.0005 \text{ inch.}$$

In this extreme case, the deflection which is due to shear is greater than that which is due to bending. If the beam were made twice as long, the bending deflection would be eight times as great, while the shear deflection would be only doubled. For beams of any considerable length relative to their cross-section, the deflection which is due to shear may be neglected.

The shearing stress in a beam is not uniformly distributed. It is possible, however, to calculate the true shear deflection for beams when the distribution of shearing stress is known. In Article 166 it will be shown by a method of work and energy, that the deflection of a beam of rectangular section may be calculated by multiplying the average unit shearing stress by the factor 1.2.

Example II

A steel cantilever, 2 inches square and 40 inches long, has a load of 240 pounds on the free end. If E_s is 12,000,000 pounds per square inch, find the shear deflection.

$$y_s = \frac{1.2 \times 60 \times 40}{12,000,000} = 0.00024 \text{ inch.}$$

Problems

1. A 2-inch by 3-inch steel beam rests on supports 12 inches apart and carries a load of 12,000 pounds midway between the supports. If E_s = 12,000,000 and E = 30,000,000 pounds per square inch, what is the shear deflection and what is the bending deflection?

   Ans. y_s = 0.0006 inch; y = 0.0032 inch.
2. Solve Problem 1 if the length of the beam is 24 inches and the load is 6,000 pounds.
3. Solve Problem 1 if the length of the beam is 60 inches and the load is 2,400 pounds.
4. The beam of Problem 1 carries a distributed load of 1,600 pounds per inch. Find the deflection which is due to shear and the deflection which is due to bending.

   Ans. y_s = 0.00048 inch; y = 0.0032 inch.

CHAPTER XI

SPECIAL BEAMS

113. Beams of Constant Strength.—In a beam of "constant strength" the unit stress in the outer fibers is the same at all sections. Since $S = \frac{Mc}{I} = \frac{M}{Z}$, the stress is constant when the section modulus varies as the bending moment. In a cantilever with a load on the end, for instance, the moment is directly proportional to the distance from the end. If the depth is constant and the width increases uniformly from the free end to the fixed end, the section modulus varies directly as the moment, and the unit stress in the outer fibers is constant. If it were not necessary to make some allowance for shear and compression at the free end, this beam would be only one-half as heavy as a uniform beam of equal strength. Even with the additional material to meet the requirements of shear and compression, a great saving in weight is secured by the use of "beams of constant strength."

114. Cantilever with Load on the Free End.—With the origin of coördinates at the free end of the cantilever, the moment at a distance x from the end is Px. (It is not necessary to consider the sign of the moment, since the unit stress depends upon the magnitude only.) If S is the allowable bending stress and Z is the section modulus,

$$Px = SZ.$$

The section modulus for a rectangular section is $\frac{bd^2}{6}$, and

$$Px = \frac{Sbd^2}{6}$$

Problems

1. A cantilever of rectangular section, with the load on the free end, has a constant depth of 6 inches. If the allowable fiber stress at all sections is 1,000 pounds per square inch, what is the equation for the breadth in terms of the load on the end and the distance from the end?

Ans. $b = \frac{Px}{6{,}000}$.

2. A cantilever of rectangular section, with the load on the free end, has a constant breadth of b units. What is the expression for the depth if the allowable stress on all sections is S?

$$Ans.\ d^2 = \frac{6Px}{Sb}.$$

3. A cantilever of rectangular section, 4 inches wide throughout, is 60 inches long, and carries a load of 800 pounds on the free end. The allowable fiber stress at all sections is 1,000 pounds per square inch. Find the depth at each 10 inches.

$$Ans.\ 800x = \frac{1{,}000 \times 4 \times d^2}{6};\ d^2 = 1.2x.$$

$x =$	10	20	30	40	50	60
$d^2 =$	12	24	36	48	60	72
$d =$	3.46	4.90	6.00	6.93	7.75	8.48

4. How does the volume of the beam of Problem 3 compare with the volume of a beam of uniform section which has the same maximum fiber stress?

5. A cantilever of rectangular section, 4 inches wide throughout, is 60 inches long, and carries a load of 600 pounds on the free end. The allowable fiber stress is 1,200 pounds per square inch. Find the depth at each 10 inches.

Ans. $x =$	10	20	30	40	50	60	inches.
$d =$	2.74	3.87	4.74	5.48	6.12	6.71	inches.

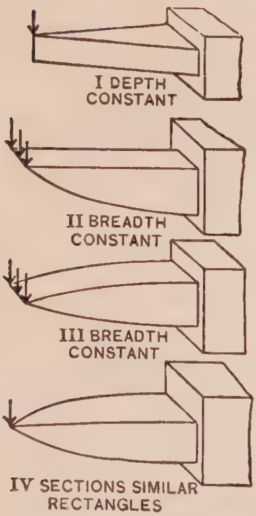


FIG. 145.—Cantilevers of constant strength.

6. A cantilever of square section is 6 feet long and carries a load of 600 pounds on the end. If the allowable unit stress is 1,000 pounds per square inch, find the dimensions at each 12 inches.

7. A cantilever of constant strength and constant breadth is 40 inches long and is 8 inches deep at the fixed end. If the load is on the free end, find the depth at each 5 inches.

Ans. $x =$	10	20	30	inches.
$d =$	4.0	5.66	6.92	inches.

8. A cantilever of constant strength, with a load on the free end, has all sections square. At the fixed end, the sections are 6 inches on a side. Find the dimensions at one-fourth, one-half, and three-fourths the length from the free end.

Figure 145 shows some cantilevers of constant strength and rectangular section. Figure 145, I, is a beam of constant depth. The breadth varies as x—the equation of a straight line. The plan is a triangle. Figure 145, II, represents a beam with breadth constant. The depth varies as the square root

of x—the equation of a parabola. One surface may be plane as in II or both may be curved as in Fig. 145, III. In any case the equation gives the total depth. Figure 145, IV, represents a cantilever in which both depth and breadth vary, all sections being similar rectangles. The equation is that of the cubical parabola.

115. Shearing and Bearing Stresses at the End.—In Fig. 145, the load P is represented at the extreme ends of the beams. Allowance must be made at the ends for the bearing and shearing stresses. For instance, in Problem 5 of Article 114, suppose

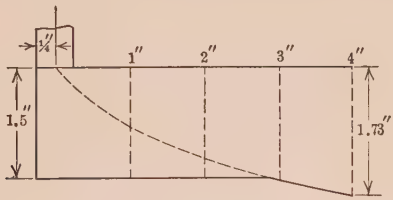


FIG. 146.

the allowable unit shearing stress to be 150 pounds per square inch. The *average* unit shearing stress in a rectangular section will be 100 pounds per square inch and the minimum area of cross-section will be 6 square inches. The depth at the end should not be less than 1.5 inches.

Suppose also that the allowable bearing stress is 300 pounds per square inch, and that the *center* of the load must be 5 feet from the wall; the bearing area must be at least 2 square inches. If the load extends the entire width of the beam, the bearing area must be 4 inches by ½ inch. The actual beam must extend at least ¼ inch beyond the center of the load. Figure 146 shows the details for these conditions. The dotted lines are the limits for the beam figured for bending only. The solid lines show the *minimum* dimensions figured for all stresses. The actual beam should be somewhat larger at the end than shown, as a great increase in safety can be secured here with practically no increase in cost and weight. Artistic appearance and convenience of construction may cause further modifications *outside* of the *minimum dimensions.*

Problems

1. Design a cantilever of constant strength and constant depth of 6 inches, which shall carry a load of 600 pounds 50 inches from the fixed end. The

allowable bending stress is 1,200 pounds per square inch; the allowable shearing stress is 120 pounds per square inch; and the allowable bearing stress is 200 pounds per square inch.

2. Design a cantilever of square section with all other conditions the same as in Problem 1.

116. Cantilever with Uniformly Distributed Load.—The only difference between a cantilever with uniformly distributed load and one with a concentrated load is in the expression for the external moment, which is $\frac{wx^2}{2}$ instead of Px.

Problems

1. A cantilever of constant strength has a rectangular section and constant breadth b. The load is uniformly distributed and is w pounds per inch of length. If S is the allowable unit stress, find the expression for the depth.

Ans. $d^2 = \frac{3wx^2}{Sb}$.

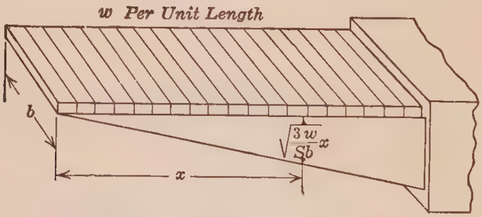


FIG. 147.—Cantilever of constant width.

2. A cantilever of constant strength, 50 inches long, has a constant breadth of 4 inches, and carries a uniformly distributed load of 240 pounds per foot. The allowable fiber stress is 1,200 pounds per square inch. Draw the front elevation.

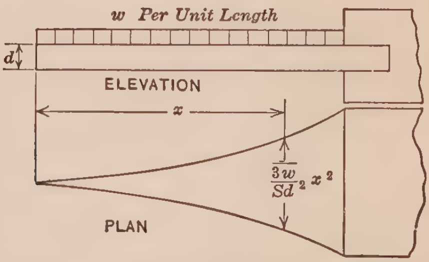


FIG. 148.—Cantilever of constant depth with uniformly distributed load.

3. A cantilever of constant strength of rectangular section is d inches deep, Fig. 148. If the load is uniformly distributed, find the expression for the breadth.

$$Ans.\ b = \frac{3wx^2}{Sd^2}.$$

4. Derive the expression for the depth of a cantilever of square section to carry a uniformly distributed load.

$$Ans.\ d^3 = \frac{3wx^2}{S}.$$

5. A cantilever of constant strength, to carry a uniformly distributed load, is 60 inches long, 6 inches wide at the fixed end, and 4 inches deep throughout. Find the width at each 12 inches.

Ans. $x =$	12	24	36	48	60 inches;
$b =$	0.24	0.96	2.16	3.84	6.0 inches.

6. If the maximum fiber stress in the beam of Problem 5 is 1,200 pounds per square inch and the maximum shearing stress is 90 pounds per square inch, for what portion of the length must the design be based on the shear? and what is the form of the end?

Ans. Plan is a triangle for the first 26.67 inches.

7. A cantilever of constant strength, which carries a uniformly distributed load, has the depth of all sections twice the breadth. The cantilever is 60 inches long, and the breadth is 6 inches at 40 inches from the free end. Find the dimensions at each 10 inches.

8. Design a cantilever of constant strength and constant breadth of 2 inches to carry a distributed load of 120 pounds per foot and a load of 480 pounds 6 inches from the free end. The beam is 5 feet long; the allowable bending stress is 1,000 pounds per square inch the allowable unit shearing stress is 90 pounds per square inch; the allowable unit bearing stress is 160 pounds per square inch.

117. Beam of Constant Strength, Supported at the Ends.—When a beam is supported at the ends and carries a single load at the middle, the problem is exactly the same as that of a cantilever of one-half the length which is pushed up by the reaction at the end. When the load is not at the middle, the portion from the load to each end is equivalent to a cantilever of constant strength. Beams with distributed loads are not so simple.

Allowance must be made for shear and bearing at the ends in all beams of this sort.

Problems

1. A cast-steel beam is made for a span of 8 feet to carry a distributed load of 2,400 pounds per foot with a maximum stress of 12,000 pounds per square inch. Find the section modulus at each 12 inches.

$$Ans.\ SZ = 9{,}600x - 100x^2;\ Z = 0.8x - \frac{x^2}{120}.$$

$x =$	12	24	36	48	inches;
$0.8x =$	9.6	19.2	28.8	38.4	
$\frac{x^2}{120} =$	1.2	4.8	10.8	19.2	
$Z =$	8.4	14.4	18.0	19.2	inches³.

2. A box girder is made of two 10-inch 25-pound channels, to which are riveted two 15-inch by ¼-inch plates. The span is 30 feet and the load is 1,800 pounds per foot. The girder is strengthened by additional ¼-inch plates extending equal distances on each side of the middle. If the allowable unit stress is 15,000 pounds per square inch, how many of these plates are required and what is the minimum length of each pair, no allowance being made for weakening of the lower plates due to rivet holes?

Shapes are not rolled as beams of constant strength, but combinations of shapes and plates, as in Problem 2, are frequently riveted together in such a way that the section modulus varies approximately as the external moment. In machinery and vehicles, where weight is important, beams of constant strength are much used. In the frames of stationary machines, these are frequently made of cast iron. In other places, cast steel or forged steel is employed. Cast-steel members, of approximately constant strength, are used in the construction of railway cars and trucks, and steel forgings in automobiles.

A tree is a vertical beam of constant strength. A bamboo rod or a wheat straw is a hollow beam of constant strength, which has a large section modulus relative to its weight.

118. Deflection of Beam of Constant Strength.—Since the moment of inertia in a beam of constant strength varies with x, the problem of finding the deflection differs from that of a uniform beam. In a beam which is symmetrical with respect to the neutral surface,

$$M = \frac{2SI}{d},$$

in which S is constant throughout the length and d may be constant or variable. In the discussions which follow, in order to distinguish between constants and variables, a constant depth is represented by a capital D and a constant breadth by a capital B.

119. Deflection of Beam of Constant Depth.—

$$M = EI\frac{d^2y}{dx^2} = \frac{2SI}{D}. \tag{1}$$

DOUBLE INTEGRATION

Dividing (1) by I:

$$E\frac{d^2y}{dx^2} = \frac{2S}{D}. \tag{2}$$

$$E\frac{dy}{dx} = \frac{2Sx}{D} + C_1. \tag{3}$$

If the direction of the x axis be so chosen that $\frac{dy}{dx} = 0$ when $x = a$, then $C_1 = -\frac{2Sa}{D}$.

$$Ey = \frac{Sx^2}{D} - \frac{2Sax}{D} + C_2. \tag{4}$$

If the origin of coördinates be so chosen that $y = 0$ when $x = a$, then $C_2 = \frac{Sa^2}{D}$, and

$$Ey = \frac{S}{D}(x^2 - 2ax + a^2) = \frac{S}{D}(a - x)^2. \tag{5}$$

At the origin, $Ey = \frac{Sa^2}{D}$. (6)

AREA MOMENTS

When the moment of inertia varies, the principle of area moments is $Ey = \int \frac{M}{I} x\,dx$; or, $Ey =$ the moment of the $\frac{M}{I}$ diagram. $\frac{M}{I} = \frac{2S}{D}$. When S and D are constant, $\frac{M}{I}$ is constant and the $\frac{M}{I}$ diagram is a rectangle. To find the deflection of the origin from the line which is tangent at the distance a from the origin,

$$Ey = \frac{2Sa}{D} \times \frac{a}{2} = \frac{Sa^2}{D}. \tag{6}$$

At a distance x from the origin, the base of the rectangle is $a - x$ and its center of gravity is $\frac{a - x}{2}$ from the point whose abscissa is x.

$$Ey = \frac{2S(a - x)}{D} \times \frac{a - x}{2} = \frac{S}{D}(a - x)^2. \tag{5}$$

Equations (5) and (6) are valid for beams of constant strength and constant depth no matter what the character of the loading. For a cantilever, $a = l$ and the moment is negative, therefore

$$Ey_{\max} = -\frac{Sl^2}{D};\ Ey = -\frac{S}{D}(l - x)^2. \tag{7}$$

CANTILEVER WITH LOAD ON THE FREE END

For a cantilever of constant depth D with a load P on the free end, $S = \dfrac{PlD}{2I_m}$, in which I_m is the maximum moment of inertia. When this value of S is substituted, Equation (7) becomes

$$Ey_{\max} = -\frac{Pl^3}{2I_m}. \tag{8}$$

The deflection at the end of a cantilever of constant strength and constant depth which is loaded at the free end, is one and one-half times as great as the deflection of a cantilever of uniform section equal to the maximum section of the constant strength beam. The volume of the constant strength beam is one-half the volume of the uniform beam.

Problem

A cantilever of constant strength and constant depth D carries a load P on the free end. The moment of inertia at the fixed end is I_m. Find the deflection at the free end by area moments without the use of Formulas (5), (6), and (7).

Ans. $\dfrac{M}{I} = -\dfrac{Pxl}{I_m x} = -\dfrac{Pl}{I_m};\ Ey_{\max} = -\dfrac{Pl^2}{I_m} \times \dfrac{l}{2} = -\dfrac{Pl^3}{2I_m}.$

CANTILEVER WITH UNIFORMLY DISTRIBUTED LOAD

For a cantilever of constant depth with uniformly distributed load, $S = \dfrac{WlD}{4I_m}$, and

$$Ey_{\max} = -\frac{Wl^3}{4I_m}, \tag{9}$$

which is twice as great as the deflection of a cantilever of uniform section equal to the maximum section of the constant

strength beam. The volume of the constant strength beam is one-third the volume of the uniform beam.

BEAM SUPPORTED AT THE ENDS WITH A LOAD P ON THE MIDDLE

The moment at the middle is $\frac{Pl}{4}$, and $S = \frac{PlD}{8I_m}$. In Equation (6), $a = \frac{l}{2}$. The deflection of the end upward from the tangent at the middle is given by

$$Ey_{\max} = \frac{Pl^3}{32I_m}, \tag{10}$$

which is one and one-half times as great as that of a beam of uniform section. This might have been taken directly from the result for the cantilever with a load on the free end.

BEAM SUPPORTED AT THE ENDS WITH UNIFORMLY DISTRIBUTED LOAD

At the middle, $M = \frac{Wl}{8}$. $S = \frac{WlD}{16I_m}$ and $a = \frac{l}{2}$;

$$Ey_{\max} = \frac{Wl^3}{64I_m}. \tag{11}$$

which is six-fifths as great as the deflection of a beam of uniform section.

120. Deflection of Rectangular Beam of Constant Breadth.—

$M = \frac{2S}{d}$, in which d is a function of x.

CANTILEVER WITH LOAD ON THE FREE END

$$d^2 = \frac{6Px}{SB}, \text{ and } \frac{M}{I} = 2S\sqrt{\frac{SB}{6P}}\,x^{-\frac{1}{2}}. \tag{1}$$

$$Ey_{\max} = \int \frac{M}{I}\,x dx = 2S\sqrt{\frac{SB}{6P}}\int_0^l x^{\frac{1}{2}}dx = \frac{2}{3} \times 2S\sqrt{\frac{SB}{6P}}\,l^{\frac{3}{2}}. \tag{2}$$

Since $\frac{M}{I_m} = 2S\sqrt{\frac{SB}{6P}}\,l^{-\frac{1}{2}}$, and $M = -Pl$,

$$Ey_{\max} = \frac{2Ml^2}{3I_m} = -\frac{2Pl^3}{3I_m}. \tag{3}$$

The deflection is twice as great as that of a beam of uniform section.

BEAM SUPPORTED AT THE ENDS WITH A LOAD AT THE MIDDLE

A beam of constant breadth and rectangular section, which is supported at the ends and loaded at the middle, is equivalent to two cantilevers.

$$Ey_{\max} = \frac{Pl^3}{24I_m}. \tag{4}$$

CANTILEVER WITH UNIFORMLY DISTRIBUTED LOAD

$$\frac{M}{I} = \frac{2S}{d};\ d^2 = \frac{3wx^2}{SB};\ \frac{M}{I} = \frac{2S}{x}\sqrt{\frac{SB}{3w}}.$$

By area moments

$$Ey_{\max} = \int \frac{M}{I}\, xdx = 2S\sqrt{\frac{SB}{3w}}\int_0^l dx = 2Sl\sqrt{\frac{SB}{3w}} \tag{5}$$

Since $\quad \dfrac{M}{I_m} = \dfrac{2S}{l}\sqrt{\dfrac{SB}{3w}}$, and $M = -\dfrac{wl^2}{2} = -\dfrac{Wl}{2}$,

$$Ey = \frac{Ml^2}{I_m} = -\frac{wl^4}{2I_m} = -\frac{Wl^3}{2I_m}, \tag{6}$$

which is four times as great as that of a beam of uniform section.

The equation of the elastic line is found (preferably by double integration) to be

$$Ey = -2S\sqrt{\frac{SB}{3w}}\left(x\log\frac{x}{l} - x + l\right). \tag{7}$$

BEAM SUPPORTED AT THE ENDS WITH UNIFORMLY DISTRIBUTED LOAD

$$M = \frac{w}{2}(lx - x^2);\ d^2 = \frac{6M}{SB} = \frac{3w(lx - x^2)}{SB}; \tag{8}$$

$$\frac{M}{I} = \frac{2S}{d} = 2S\sqrt{\frac{SB}{3w(lx - x^2)}}. \tag{9}$$

$$Ey_{\max} = \int \frac{M}{I}\, xdx = 2S\sqrt{\frac{SB}{3w}}\int \frac{xdx}{\sqrt{lx - x^2}}. \tag{10}$$

$$\int \frac{xdx}{\sqrt{lx - x^2}} = \int \frac{xdx}{\sqrt{\left(\frac{l}{2}\right)^2 - \left(\frac{l}{2} - x\right)^2}} = -\frac{l}{2}\int \frac{dz}{\sqrt{\left(\frac{l}{2}\right)^2 - z^2}} + \int \frac{zdz}{\sqrt{\left(\frac{l}{2}\right)^2 - z^2}}; \tag{11}$$

where $z = \frac{l}{2} - x$; $dz = -dx$.

$$Ey_{\max} = 2S\sqrt{\frac{SB}{3w}}\left[-\frac{l}{2}\sin^{-1}\frac{2z}{l} - \sqrt{\left(\frac{l}{2}\right)^2 - z^2}\right]_{z=\frac{l}{2}}^{z=0} \qquad (12)$$

$\left(\text{When } x = 0,\ z = \frac{l}{2};\ \text{when } x = \frac{l}{2};\ z = 0\right)$.

$$Ey_{\max} = Sl\sqrt{\frac{SB}{3w}}\left(\frac{\pi}{2} - 1\right). \qquad (13)$$

Since
$$S = \frac{3wl^2}{4BD^2},$$

$$Ey_{\max} = \frac{3wl^4}{8BD^3}\left(\frac{\pi}{2} - 1\right) = \frac{wl^4}{32I_m}\left(\frac{\pi}{2} - 1\right) = \frac{6.85wl^4}{384I_m}. \qquad (14)$$

121. Deflection of Beam of Constant Strength with All Sections Similar.—$I = kc^4$, in which k is a constant which depends upon the geometry of the sections, and c is the distance from the neutral axis to the outer fiber. $Z = \frac{I}{c} = kc^3$.

$$S = \frac{M}{Z} = \frac{M}{kc^3};\ c^3 = \frac{M}{kS};$$

$$\frac{M}{I} = \frac{M}{kc^4} = M^{-1/3}k^{1/3}S^{4/3}. \qquad (1)$$

DEFLECTION OF A CANTILEVER WITH LOAD ON THE FREE END

$$M = Px, \text{ and } \frac{M}{I}x = P^{-1/3}k^{1/3}S^{4/3}x^{2/3};$$

$$\int_0^l \frac{M}{I}\,x dx = \frac{3P^{-1/3}S^{4/3}k^{1/3}l^{5/3}}{5} \qquad (2)$$

Since $S = \frac{Pl}{kc_m^3}$, in which c_m is the maximum value of c;

$$Ey_{\max} = \frac{3Pl^3}{5kc_m^4} = \frac{3Pl^3}{5I_m}, \qquad (3)$$

which is nine-fifths as great as the deflection of a uniform beam.

DEFLECTION OF A CANTILEVER WITH UNIFORMLY DISTRIBUTED LOAD

$$M = \frac{wx^2}{2};\ \frac{M}{I}xdx = w^{-1/3}2^{1/3}k^{1/3}S^{4/3}x^{1/3}dx;$$

$$\int_0^l \frac{M}{I}\,x dx = \frac{3w^{-1/3}2^{1/3}k^{1/3}S^{4/3}l^{4/3}}{4}. \qquad (4)$$

Since $$S = \frac{wl^2}{2kc_m^3},\quad S^{4/3} = \frac{w^{4/3}l^{8/3}}{2^{4/3}k^{4/3}c_m^4},\ \text{and}$$

$$Ey_{\max} = \frac{3wl^4}{8kc_m^4} = \frac{3wl^4}{8I_m} = \frac{3Wl^3}{8I_m} \tag{5}$$

which is three times as great as the deflection of a beam of uniform section.

122. Beams of Two or More Materials.—Beams are frequently made of two or more materials which have different moduli of elasticity. The most common types are combinations of timber and steel, or of concrete and steel.

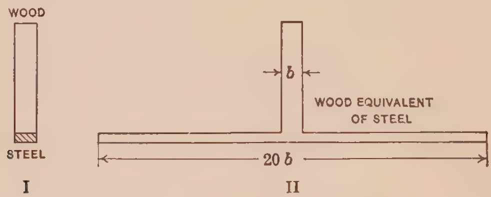


FIG. 149.—Beam of two materials.

Figure 149 shows a steel plate which is bolted to the bottom of a wooden beam. The position of the neutral surface depends upon the ratio of the moduli of elasticity of the two materials. If the modulus of the steel is 30,000,000 pounds per square inch and that of the wood is 1,500,000 pounds per square inch, the unit stress in the steel at a given unit deformation is 20 times as great as the unit stress in the wood.

The location of the neutral axis for Fig. 149 may be computed on the assumption that the density of the steel is 20 times as great as the density of the wood; or, for purpose of calculation, the steel may be replaced by a wooden strip which is 20 times as wide and has the same thickness. Figure 149, II, illustrates this substitution.

Example

A 4-inch by 6-inch wooden beam has a steel plate 1 inch wide and ½ inch thick fastened to the lower surface. Find the neutral axis and the maximum fiber stress in each material if the modulus of elasticity of the steel is 20 times as great as that of the wood, and the bending moment is 30,000 inch-pounds.

The steel may be replaced by a wooden strip 20 inches wide and ½ inch thick. To get the distance of the center of gravity from the bottom of the wood,

$$\bar{y} = \frac{24 \times 3 - 10 \times \frac{1}{4}}{34} = 2.04 \text{ inches.}$$

To get the moment of inertia of the equivalent wooden section about the common surface.

$$\frac{4 \times 6^3}{3} = 288,$$

$$\frac{20 \times (\frac{1}{2})^3}{3} = 0.83,$$

$$I = 288.83.$$

$$I_0 = 288.83 - 34 \times 2.04^2 = 147.34 \text{ inches}^4.$$

To get the unit stress in the top fibers of the wood

$$S = \frac{30{,}000 \times 3.96}{147.34} = 806 \text{ pounds per square inch.}$$

In the bottom steel fibers

$$S = \frac{30{,}000 \times 2.54 \times 20}{147.34} = 10{,}344 \text{ pounds per square inch.}$$

The result for steel is multiplied by 20 because the moment of inertia used was calculated on the assumption that the steel was replaced by wood.

Problems

(Use E for steel 20 times E for timber in these problems)

1. A 4-inch by 4-inch timber beam has a 4-inch by ½ inch steel plate on the lower surface and a 2-inch by 1-inch plate on the upper surface. Find the neutral axis of the combination. What is the maximum fiber stress in the steel when that in the wood is 600 pounds per square inch.
Ans. Neutral axis, 2.10 inches above bottom of timber; fiber stress in steel, 16,571 pounds per square inch

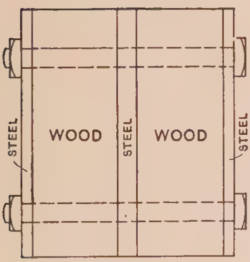


FIG. 150.—Flitched beam.

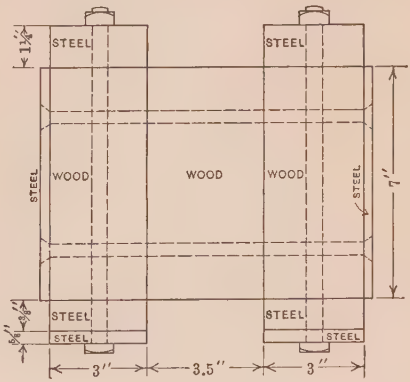


FIG. 151.—Armored wooden beam.

2. A 6-inch by 6-inch timber beam, 10 feet long, has a 6-inch by ½-inch steel plate on the top and bottom surfaces. Find the unit stress in the steel when a load of 9,000 pounds is put on the middle.
Ans. 13,720 pounds per square inch.

Figure 150 is called a flitched beam. It is built up of alternating wooden beams and steel plates, all fastened together by a

few bolts. When the vertical depth of the wooden beams and steel plates is the same, the bolts transmit no shear below the elastic limit of the steel or wood. With vertical loads, parallel to the plane of the plate, the steel is not efficiently used in this type of beam.

Flitched beams were once used to some extent, but very little at present. Steel I-beams are usually preferable. Wooden beams are frequently bolted to the web of an I-beam. This is generally done for convenience in attaching woodwork rather than for reinforcing the beam.

When steel is fastened to the top and bottom of a wooden beam, it is used efficiently. The combination is equivalent to an I-beam section, the timber acting as the web and the steel as the flange.

Such combinations are not used in structures, but are employed in vehicles and some classes of machinery. Figure 151 is a section of the lower end of the dipper handle of a steam shovel. It consists of an oak beam made of three pieces (the middle piece is omitted at the upper end) with heavy steel plates at the top and bottom and thin plates on the sides. If the beam were made of the steel plates alone, without the timber, the compression side would be liable to buckle. Kinks and dents from blows and other rough usage would increase this danger. The high elastic limit of well-seasoned hard wood makes it desirable in locations of this kind.

123. Reinforced-concrete Beams.—Reinforced concrete represents another form of combination beam. A reinforced-concrete beam has steel rods embedded in the concrete near the surface in the tension side. Sometimes both tension and compression sides are reinforced. These rods may be ordinary round or square steel bars. Usually they are corrugated or otherwise deformed, or made of cable or twisted square bar. Such deformed or twisted bars are better fitted to transmit the stress from the concrete, since they do not slip when the grip of the concrete is weakened.

Figure 152 represents a portion of a reinforced-concrete beam 8 inches by 11 inches in cross-section. The reinforcement consists of three rods with centers 1 inch from the bottom of the beam. The photograph, Fig. 204, shows a beam of this size after failure.

In working out the theory of concrete beams, it is customary to regard the steel as taking all the tension. If the unit stresses

are kept low, the concrete on the tension side of the neutral axis does exert some tensile stress, but at loads of less than one-third of the ultimate strength of the beam, fine cracks form in the tension side and tests show that the steel takes practically all the tension at larger loads.

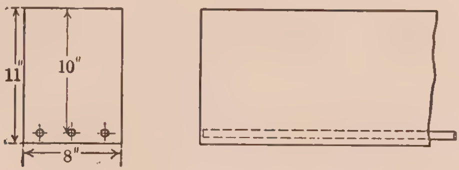


FIG. 152.—Reinforced concrete beam.

The *per cent. of reinforcement* in a beam is calculated by dividing the area of the steel by the area of the beam section above the center of the steel. In Fig. 152 the beam is regarded as an 8-inch by 10-inch section; the inch of concrete below the center of the rods is considered as simply protecting the steel. With three $\frac{5}{8}$-inch rods, each of which has a cross-section of 0.307 square inch, the reinforcement in the beam of Fig. 152 is $0.921 \div 80 = 0.0115 = 1.15$ per cent. While it is customary to speak of the per cent. of reinforcement, when used in formulas it is expressed as a ratio.

Elaborate formulas have been proposed for the calculation of reinforced-concrete beams. Some of these formulas assume that the compression curve of concrete is a parabola, *which it is not.* The form and the constants of the compression curve vary greatly with the materials, the proportions, the care in mixing, the age, and the stresses to which it has been subjected. The modulus of elasticity is lowered greatly by slight overloads. For these reasons there is little use for great refinement of calculation unless the computer is provided with carefully determined compression curves of the actual concrete under consideration, and it is now customary to work on the assumption that the compression curve is a straight line.

A Joint Committee from The American Society of Civil Engineers, The American Society for Testing Materials, The American Railway Engineering Association, and The Association of American Portland Cement Manufacturers has prepared a report on "Concrete and Reinforced Concrete" and has recommended certain formulas and constants. In the articles which follow,

the important formulas will be given in the form proposed by the Joint Committee, and with the same symbols except that s will be used for unit stress in place of f. The constants given in Table XI agree with the report in round numbers.*

124. Location of the Neutral Axis.—The line OF of Fig. 153 represents the compressive stress. The depth from the extreme compressive fibers to the center of the reinforcement is d and the

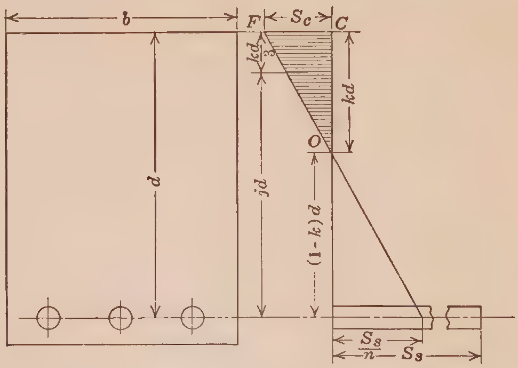


FIG. 153.—Stress in rectangular concrete beam.

distance from the neutral surface to the extreme fibers is kd, in which k is a fraction less than unity. The distance from the neutral surface to the center of the reinforcement is $(1 - k)d$. The ratio of the modulus of elasticity of the steel to that of the concrete is represented by n;

$$n = \frac{E_s}{E_c}. \tag{1}$$

If the modulus of elasticity of the steel is 30,000,000 and that of the concrete in compression is 2,000,000 pounds per square inch the value of n is 15. This is the value which is recommended for general use. However, when tests show the ultimate strength of the concrete to be between 2,200 pounds and 2,900 pounds per square inch, the Joint Committee recommends that n should be 12; and for concrete stronger than 2,900 pounds per square inch, the committee recommends that n should be 10.

The unit deformation at the center of the reinforcement is to the unit deformation in the outer fibers of the concrete as $1 - k$ is to k. Since the modulus of the steel is n times the

* *Proceedings of the American Society for Testing Materials*, vol. XIII, 1913.

modulus of the concrete, the unit stress at the center of the reinforcement is

$$s_s = \frac{n(1-k)}{k} S_c, \tag{2}$$

in which s_s is the unit tensile stress at the center of the steel reinforcement, and S_c is the maximum unit compressive stress in the concrete. The tensile stress at the center of the reinforcement is the average tensile stress, and it is assumed that the resultant tensile stress is at the center of the section. (The error is negligible.) The area of the concrete in compression in a rectangular section is bkd, and the average unit stress over this area is $\frac{S_c}{2}$.

$$\text{Total compressive stress} = \frac{S_c kbd}{2}. \tag{3}$$

$$\text{Total tensile stress in steel} = \frac{AnS_c(1-k)}{k}, \tag{4}$$

in which A is the area of the steel. The ratio of the area to the steel to the area of the concrete is represented by p;

$$p = \frac{A}{bd}; \; A = pbd, \tag{5}$$

As the concrete below the neutral surface is not regarded as taking any of the tensile stress, the total tension in the steel equals the total compression in the concrete. Equating (3) and (4) and substituting for A,

$$\frac{S_c kbd}{2} = \frac{S_c pbdn(1-k)}{k}, \tag{6}$$

$$k^2 = 2pn(1-k), \tag{7}$$

$$k^2 + 2pnk - 2pn = 0, \tag{8}$$

$$k = \sqrt{2pn + (pn)^2} - pn. \tag{9}$$

Problems

1. If the modulus of the steel be taken as 15 times that of the concrete and the area of the steel is 1 per cent. of the total area bd, find the distance of the neutral axis from the extreme compression fibers. *Ans.* $k = 0.418$.

2. Solve Problem 1 for a reinforcement of 1.2 per cent. and for 1.6 per cent. *Ans.* 0.446, 0.493.

125. Relative Unit Stresses in Concrete and Steel.—When the location of the neutral axis has been determined by means

of Equation (9) of the preceding article, or by experiment, the relative values of the average unit compressive stress in the concrete and the average unit tensile stress in the steel may be computed from the relation that the total tension in the steel is equal to the total compression in the concrete. In Problem 1 of the preceding article, for instance, since the area of concrete in compression is 0.418 bd and the area of the steel in tension is $0.01bd$, the average unit stress in the concrete is $^{10}/_{418}$ as great as the average unit stress in the steel. If the average unit stress in this steel is 12,000 pounds per square inch, the average unit compressive stress in the concrete is 287 pounds per square inch, and the maximum stress in the extreme fibers is 574 pounds per square inch.

The unit compressive stress in the extreme fibers may also be computed from the distances from the neutral axis and the ratio of the two moduli of elasticity. From Equation (2) of the preceding article.

$$S_c = \frac{ks_s}{(1-k)n} \tag{1}$$

TABLE XI.—ALLOWABLE UNIT COMPRESSIVE STRESSES IN EXTREME FIBERS OF CONCRETE BEAMS, IN POUNDS PER SQUARE INCH

Aggregate	1:2:4	1:3:6
Gravel or hard limestone or sandstone..........	650	425
Soft limestone or sandstone....................	500	325
Cinder....................................	200	125

Problems

1. In Problem 2 of the preceding article, calculate the unit stress in the extreme fibers when the average unit tensile stress in the steel is 12,000 pounds per square inch. *Ans.* 645 and 778 pounds per square inch.
2. Solve Problem 1 if the allowable unit stress in the steel is 16,000 pounds per square inch.

With a unit stress in the steel of 12,000 pounds per square inch, the unit compressive stress in the concrete with 1.6 per cent. reinforcement is above the allowed value for 1:2:4 concrete for the best material ordinarily used. If the allowable unit stress

in the steel is 16,000 pounds per square inch (which is the maximum recommended by the Joint Committee) even 1 per cent. of reinforcement gives a unit stress in the concrete of over 700 pounds per square inch. In order to use the steel efficiently in a beam of rectangular section, it is necessary to have a richer mix than 1:2:4 or to keep the reinforcement below 1 per cent.

Problem

3. If $n = 15$ and $p = 0.008$ what will be the unit stress in the steel when the unit stress in the outer fibers of the concrete, calculated on the assumption that the compression curve is a straight line, is 650 pounds per square inch. *Ans.* 15,600 pounds per square inch.

126. The Resisting Moment.—The resultant compressive stress is at the center of gravity of the triangle CFO of Fig. 153. The resultant tensile stress is regarded as being at the center of the reinforcement, therefore the arm of the resisting moment is $\left(1 - \frac{k}{3}\right)d$. The term $\left(1 - \frac{k}{3}\right)$ is represented by the single letter j.

$$\text{Resisting moment arm} = \left(1 - \frac{k}{3}\right)d = jd. \qquad (1)$$

Problems

1. What is the resisting moment arm in Problem 1 of Article 124? *Ans.* $jd = 0.86d$.

The resisting moment is either total stress multiplied by the moment arm,

$$M = \frac{S_c jkbd^2}{2} = s_s Ajd; \qquad (2)$$

$$S_c = \frac{2M}{jkbd^2}; \qquad (3)$$

$$s_s = \frac{M}{Ajd}. \qquad (4)$$

(In the following problems $n = 15$, $S_c = 650$ lb./in.^{2})

2. A reinforced-concrete beam for a span of 15 feet is 10 inches wide and 12 inches deep to center of reinforcement. The reinforcement consists of three deformed bars, each having a cross-section of 0.39 square inch. The beam weighs 125 pounds per linear foot. What is the maximum safe load on the middle, based on the compressive strength of the concrete? What is the unit tensile stress in the steel at this load.

Ans. $M = 167{,}000$ inch-pounds; maximum safe load 2,770 pounds.

3. In Problem 2 find the unit stress in the steel by dividing the moment by the resisting arm to get the total tension, and then dividing by the area of the steel. *Ans.* 13,800 pounds per square inch.

4. Design a reinforced-concrete beam for a span of 20 feet to carry a load of 800 pounds per foot including its own weight, using 1 per cent reinforcement.

An approximate value of the resisting moment may be computed from the expression:

$$M = 0.8d \times As_s.$$

The moment arm is always a little greater than $0.8d$ and the total tensile stress in the reinforcement is As_s. Of course, if the percentage of reinforcement is too great, the compressive stress in the concrete will be too high.

127. Steel Ratio.—It was shown in Article 125 that when the percentage of reinforcement is too great, the concrete stress will exceed its allowable value before the steel is fully stressed. The ratio of the steel area to total area may be found for any allowable unit stresses. From the equality of the total tensile and compressive stress,

$$\frac{S_ckbd}{2} = s_sA, \tag{1}$$

from which

$$k = \frac{2s_sA}{S_cbd} = \frac{2s_sp}{S_c}. \tag{2}$$

From Equation (8) of Article 124,

$$k^2 + 2pnk = 2pn. \tag{3}$$

Eliminating k between Equations (2) and (3),

$$\frac{4s_s^2p^2}{S_c} + \frac{4s_sp^2n}{S_c} = 2pn; \tag{4}$$

$$p = \frac{n}{2\frac{s_s}{S_c}\left(\frac{s_s}{S_c} + n\right)} = \frac{1}{2\frac{s_s}{S_c}\left(\frac{s_s}{nS_c} + 1\right)}. \tag{5}$$

Problem

Find the steel ratio if the allowable unit compressive stress in the concrete is 600 pounds per square inch, the allowable tensile stress in the steel is 15,000 pounds per square inch and the ratio of the modulus of elasticity of the steel to that of the concrete is 15. *Ans.* $p = 0.0075$.

See Report of the Joint Committee on "Concrete and Reinforced Concrete," *Proceedings of The American Society for Testing Materials*, 1913, page 278.

CHAPTER XII

BENDING COMBINED WITH TENSION OR COMPRESSION

128. Transverse and Longitudinal Loading.—It often happens that a beam is subjected to a direct tension or compression in the direction of its length and a transverse force producing a bending moment. The unit stress at any point in a given section is the sum of the direct stress and the bending stress at that point. For example, suppose a 4-inch by 4-inch post stands vertical and supports a load of 4,000 pounds at the top. The direct compressive stress is 250 pounds per square inch. Suppose this post is fixed at the bottom (Fig. 154), and that a horizontal push of 200 pounds is applied 2 feet from the bottom. This transverse force produces a tensile stress of 450 pounds per square inch in the outer fibers at the bottom on the side of the push and a compressive stress of the same magnitude in the opposite side. The resultant stress is 700 pounds per square inch in the one side and 200 pounds per square inch in the other.

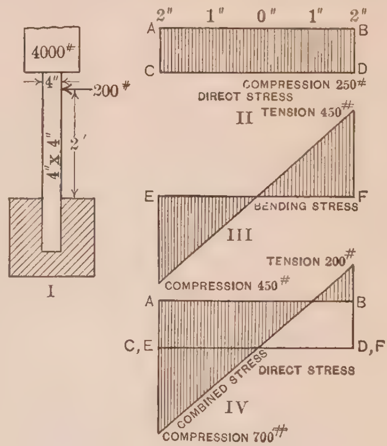


Fig. 154.—Post with compression and bending.

Figure 154, IV, shows the distribution of stress, with compression represented by a negative ordinate and tension by a positive ordinate. Figure 154, II, shows the compression alone which is due to the direct load of 4,000 pounds. Figure 154, III, shows the stress which is caused by the bending. This stress is 450 pounds per square inch compression on the left side and 450 pounds per square inch tension on the right side. At the middle of the section, the bending stress is zero. In Fig. 154,

IV, the stresses are combined. The line EF, which is the zero line for the bending stress, is placed on the line CD, which represents the compressive stress in II. The combined unit compressive stress on the left side is $250 + 450 = 700$ pounds per square inch. On the right side the combined stress is 450 tension minus 250 compression, or 200 pounds per square inch tension. The unit stress is zero at $8/9$ inch from the right side of the post.

$$\text{Unit stress} = \frac{P}{A} + \frac{Mv}{I}, \qquad \text{Formula XXIV.}$$

in which P is the total load parallel to the length of the beam and M is the bending moment from any source whatever. Since v has the positive sign on one side of the neutral axis and the negative sign on the other side, the second term may be positive or negative, according to the position.

Problems

1. A wooden post, 6 inches square and 5 feet high, carries a load of 7,200 pounds on the top and is pushed south by a horizontal force of 225 pounds applied 1 foot from the top. Find the stress at the north and south faces at the bottom.
Ans. 500 pounds per square inch compressive stress at the south; 100 pounds per square inch tensile stress at the north.
2. At what distance from the top is the tensile stress zero on the north face? *Ans.* 44 inches.
3. At what distance from the north face at the bottom is the unit stress the post of Problem 1 equal to zero. Solve by Formula XXIV and check by interpolation from the answers of Problem 1.
4. A rectangular pier is 3 feet by 4 feet and is 40 feet high. It weighs 120 pounds per cubic foot. What wind pressure, in pounds per square foot, uniformly distributed over one entire face, will overthrow this pier if it carries no load except its own weight and is not able to resist tension?
5. A post, 6 inches square and 5 feet long, is fixed at the lower end and carries a load of 7,200 pounds at the middle of the upper end. The post makes an angle of 5 degrees with the vertical in the plane of one pair of faces. Find the maximum and minimum unit stress at the bottom.
Ans. $199 \pm 1{,}046$ pounds per square inch.

129. Eccentric Loading.—Figure 155 represents a rigid bar G supported by three equal rubber bands (or springs) which are symmetrically placed and suspended from a rigid horizontal support. Each of the bands is stretched the same amount and the bar hangs in a horizontal position. Figure 155, II, shows the same bar with a load P at the middle. The rubber bands are

equally stretched and the bar remains in a horizontal position. If the load P be moved to the right, as in Fig. 155, III, the middle band receives the same elongation as in the preceding case, while the left band is elongated less and the right band more. If the load be moved still farther to the right, a place is reached where the left end is elevated above the position which it occupied before the load was applied, so that no load whatever comes on the left band. If, instead of the rubber bands, helical springs were used, the spring on the side away from the load would come into compression when the load is shifted a considerable distance from the middle.

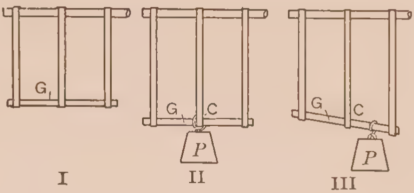


FIG. 155.—Eccentric loading of rubber bands.

Instead of the rubber bands of Fig. 155, a continuous sheet of rubber or metal might be used. If such a sheet is fastened to a rigid body at the top, and to another rigid body at the bottom, and if a load is then applied to the lower rigid body considerably to one side of the center of the sheet, the sheet is elongated on that side and shortened or buckled on the other. A similar result obtains when a compressive load is applied to a body. Figure 156, I, shows a block of soft rubber with the load central, and Fig. 156, II, shows the effect of moving this load a little to one side.

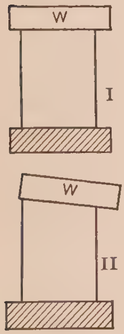


FIG. 156. Compression with direct and with eccentric loading.

Figure 155, III, shows that the effect of the eccentric load is a translation downward, of the same magnitude as that caused by the central load in II, together with a rotation about the bottom of the middle band as an axis. Equilibrium occurs when the moment of the load P about the lower end of the middle band is equal to the moment of the excess of tension in the right band plus the moment of the deficiency of tension in the left band. Suppose that the bands are 4 inches apart, and that a load of 1 pound at the middle stretches all three bands 0.4 inch. One pound will stretch a single band 1.2 inches. Now move the load of 1 pound 2 inches to the right of the middle. The moment with respect to the middle is 2 inch-

pounds, which may be balanced by a load of 0.5 pounds 4 inches from the axis. The tension in the right band is 0.25 pound more and the tension in the left band is 0.25 pound less than that in the middle band. The total tension is 1/12 pound in the left band, 1/3 pound in the middle band, and 7/12 pound in the right band. These may be checked by moments around either the left band, the right band, or about any other axis whatever. If the load is moved more than 3 inches from the middle, the tension in the left band becomes less than it was before this 1-pound load was applied.

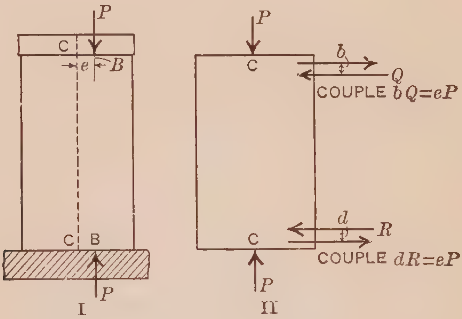


FIG. 157.—Block with eccentric loading.

Figure 157, I, shows a block, which is subjected to a load P at a distance e from its axis. A force along any line may be replaced by an equal force along any parallel line and a couple the moment of which is equal to the product of the force by the distance between the lines.*

In Fig. 157, the force P at the top, at a distance e from the axis, may be replaced by a force P along the axis and a clockwise couple of moment $e \times P$. The reaction at the bottom may likewise be regarded as equivalent to a reaction P along the axis and a counter-clockwise couple of moment $e \times P$. These equal couples are shown in Fig. 157, II. An eccentric load may be regarded as equivalent to a load along the axis combined with a bending moment, which is the product of the load multiplied by the eccentricity.

Figure 158 shows examples of large eccentricity in which the existence of direct stress combined with bending stress is almost self-evident. The portion above G in Fig. 158, I, may be treated as a free body. A vertical resolution shows that the vertical

* See any textbook of Mechanics.

reaction across the section at G is equal to the load P. The moment of the load P about an axis perpendicular to the plane of the paper through the center of the section at G is $e \times P$.

Figure 158, II, shows the deflection which is caused by compression. The deflection of the bar increases the eccentricity. Figure 158, III, shows the effect of tension. The deflection of the bar reduces the eccentricity.

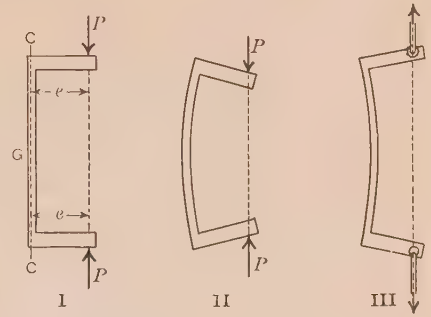


FIG. 158.—Large eccentricity.

Since an eccentric load is equivalent to a central load and a couple of moment eP, the unit stress at a distance v from the center of gravity of any section is

$$s = \frac{P}{A} \pm \frac{Mv}{I}. \qquad \text{Formula XXIV.}$$

At the outer fibers,

$$S = \frac{P}{A} \pm \frac{Mc}{I} = \frac{P}{A} \pm \frac{M}{Z}.$$

In these formulas, $M = e \times P$ is the moment of the load about the axis through the center of gravity of the section, v is the distance from the center of gravity of the section to any given fiber, and c is the distance from the center of gravity of the section to the extreme outer fiber.

Formula XXIV for eccentric loading assumes that the section at which the load is applied remains plane. This condition is approximately fulfilled when the load is applied through a relatively rigid plate or cap stone to a body which is comparatively elastic. When the load is concentrated, the maximum stresses are usually greater than those given by the formula. If the block under stress is of some length, the sections near the middle are practically plane and the formula applies with greater accuracy.

The derivation of Formula XXIV assumes that E is constant. This assumption limits the formula to stresses below the elastic limit. Since the stress on one side is greater than on the other, it will reach the elastic limit before the other and cause a shifting

of the neutral axis for the bending stresses. The error caused by this displacement of the neutral axis is greater than in a beam subjected to bending alone.

Subject to a correction for curvature, (see Chapter XVIII) formula XXIV applies approximately to hooks.

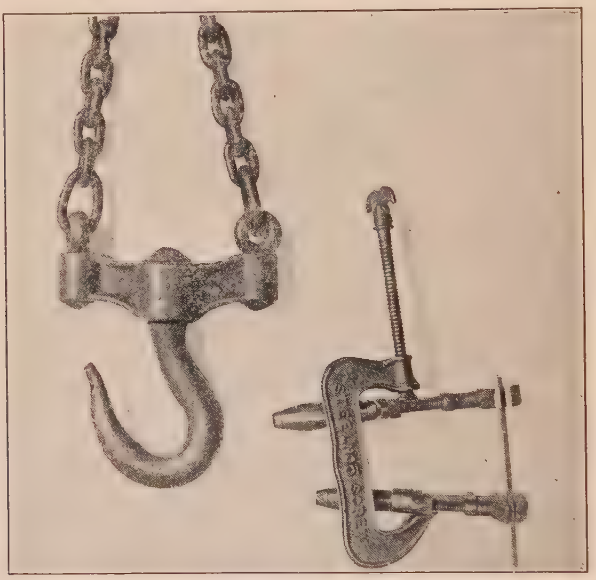


Fig. 159.—Eccentric loading.

Some examples of compression combined with bending are shown in Fig. 159. The forces P are applied to the wrenches by means of the screw clamp. The wrenches as cantilevers transmit the direct compression and the bending moments to the bar. The experiment may be performed with two wrenches and a metal or wooden bar with the forces applied to the wrenches by the hands instead of the clamp. The bar will bend as in Fig. 158, II, when the forces are toward each other and as in Fig. 158, III, when the forces are away from each other.

The clamp of Fig. 159 is subjected to *tension* and bending. The eccentricity is the distance from the center of the screw to the center of gravity of any section. In a hook, the line of the load joins the shank with the point which is immediately below it when loaded. This is, of course, the point in the concave portion which is farthest from the shank. The eccentricity

is the distance of the center of gravity of any section from this load line.

Problems

1. A 6-inch by 6-inch short block is subjected to a load of 7,200 pounds. The line of application of this load lies in a plane parallel to a pair of the faces at a distance of ½ inch from the axis of the block. Find the maximum and minimum compressive stress.

Ans. $S = 200 + 100$ pounds.

2. Solve Problem 1 for an eccentricity of 1 inch.

Ans. 400 pounds; 0 pounds.

3. Solve Problem 1 for an eccentricity of 2 inches.

4. A solid circular rod, 2 inches in diameter, is subjected to a pull of 18,000 pounds along a line 0.2 inch from the axis. Find the maximum and minimum unit stress. *Ans.* 10,313 pounds; 1,146 pounds.

5. Solve Problem 4 if the eccentricity is 0.3 inch.

Ans. 12,605 tension; 1,146 compression.

6. Solve Problem 5 if the rod is hollow with inside diameter of 1 inch.

7. A block b wide and d thick, of rectangular section, has the load so placed that the unit stress in the outer fibers on one side is zero. If the line of load is in the plane of symmetry parallel to the faces of breadth d, what is the eccentricity?

Ans. $\frac{d}{6}$.

8. What eccentricity in a solid circular section of radius a will make the unit stress on one side zero?

Ans. $e = \frac{a}{4}$.

9. A hollow circular cylinder of outside radius a and inside radius b is so loaded that the unit stress on one side is zero. What is the eccentricity?

Ans. $e = \frac{a^2 + b^2}{4a}$.

10. A solid wall has the resultant load 2 feet from the front edge. The load is 12 tons per running foot. Assuming that the load is so distributed that the top remains plane, find the unit stress in tons per square foot at the front edge if the breadth of the wall is 4 feet, 6 feet, 8 feet, 10 feet.

Ans. 3, 4, 3.75, 3.36 tons per square foot compression.

11. In a hook of circular section the distance from the center of gravity of the section to the line of the load is 3 inches. The load is 1,600 pounds and the diameter of the section is 2 inches. Using Formula XXIV, find the *approximate* value of the maximum tensile and compressive stress.

Ans. 6,621 pounds per square inch tension.
5,602 pounds per square inch compression.

130. Maximum Eccentricity without Reversing Stress.—A brick pier laid in lime mortar has no tensile strength, and the tensile strength of masonry laid in cement mortar is uncertain.

For this reason, the load on a masonry pier or wall should always be so placed that the stress over the entire section shall be compressive. Problem 7 of the preceding Article showed that a load on a rectangular section at a distance from the center greater than one-sixth the width of the section in that direction produces a negative stress in the opposite side. For this reason, it is a rule of architects and engineers that **the resultant load shall not fall outside the middle third of a wall or of a rectangular pier or column.** In Fig. 160, if the load, perpendicular to the plane of the paper, passes through the line BD, which is midway between the two sides of width d, it must lie inside the middle third of this line. If the load passes through the line FG, it must pass between the points C_1C_1. C_1C_1 is one-third of FG and CC is one-third of BD.

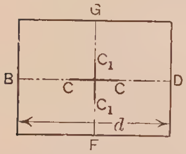


FIG. 160.—Maximum eccentricity on principal axes of rectangular section.

From Problem 8 of the preceding article, it is evident that the load on a solid pier of circular section must not fall outside a circle whose diameter is one-fourth that of the pier. Problem 9 shows that with a hollow pier the eccentricity may be greater without reversing the stress. For materials which are weak in tension, therefore, a hollow pier may be stronger than a solid pier of the same outside dimensions.

For any section, the maximum eccentricity without reversing stress may be computed from Formula XXIV. Since the fibers under zero stress are on the side of the center of gravity opposite the load, the form with the negative sign is used.

$$S = 0 = \frac{P}{A} - \frac{Mc}{I}; \tag{1}$$

$$\frac{P}{A} = \frac{Mc}{I} = \frac{Pec}{I} = \frac{Pec}{Ar^2}; \tag{2}$$

in which r is the radius of gyration with respect to the axis through the center of gravity of the section.

$$e = \frac{r^2}{c}. \tag{3}$$

Problems

1. A hollow pier is 24 inches square on the outside and 16 inches square inside. How great may be the eccentricity on a line through the center parallel to the sides without reversing the stress?

Ans. $e = \dfrac{832}{12 \times 12} = 5\frac{7}{9}$ inches.

2. In a solid pier 12 inches square, how far may the resultant be placed from the center of gravity of the section if it is on a line through the center parallel to two faces? *Ans.* 2 inches.
3. Solve Problem 2 if the load is on a diagonal. *Ans.* 1.41 inches.
4. A square section of side b has the resultant load at a point C, the coördinates of which are (x, y), Fig. 161, I. Show that when the unit stress at F is zero, the position of C satisfies the equation

$$6x + 6y = b.$$

SUGGESTION.—The moment of inertia of a square section being the same for all axes through the center, the rotation will be about the axis OE perpendicular to OC. The distance of the extreme fibers at F from this axis is equal to EB.

The distance

$$EB = \frac{b}{2}(\cos\theta + \sin\theta).$$

$$I = \frac{b^4}{12}.$$

For zero stress at the corner, F,

$$e(\cos\theta + \sin\theta) = \frac{b}{6}.$$

$$x + y = \frac{b}{6}.$$

131. Resultant Load Not on a Principal Axis.—In all the problems of the preceding articles, the resultant load fell on one principal axis and rotation took place about the other principal

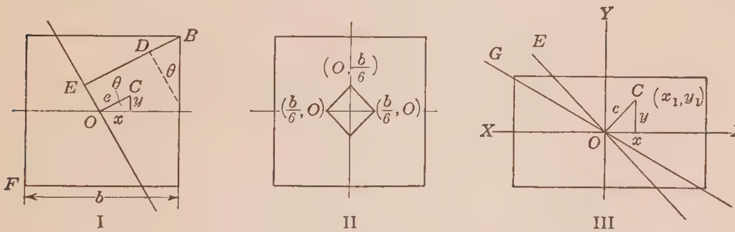


FIG. 161.—Eccentric load not on principal axis.

axis. When the section is a circle, a square, or any other regular polygon, the moment of inertia is the same for all axes through the center of gravity, and any such axis is a principal axis. In other sections, when the load does not fall on a principal axis, the axis of rotation is not the line OE (Fig. 161, III) normal to OC, but is some other line OG which lies between OE and the axis for which I is the minimum.

To find the bending stress caused by an eccentric load, the eccentricity is resolved into two components parallel to the two

principal axes of inertia, and the stress is computed separately for each component. In Fig. 161, III, the force is applied perpendicular to the plane of the paper at the point C. The eccentricity is OC, and the components of the eccentricity are x and y. The moment about the X axis is Py. The bending stress of this moment at a point whose coördinates are x_1, y_1 is $\frac{Pyy_1}{I_x}$. The moment about the Y axis is Px, and the bending stress of this moment at the point (x_1,y_1) is $\frac{Pxx_1}{I_y}$.

The combined unit stress at any point (x_1,y_1) when the load is applied at the point (x,y) is

$$s = \frac{P}{A} + \frac{Pxx_1}{I_y} + \frac{Pyy_1}{I_x}. \tag{1}$$

If x and x_1 have the same sign, the second term of the second member of Equation (1) is positive, and if y and y_1 have the same sign, the third term is positive.

Example

A rectangular block is 12 inches long, measured from east to west, and 10 inches wide, from south to north. It is subjected to a load of 3,600 pounds, which is applied 2 inches from the east edge and 2 inches from the north edge. Find the unit stress at each corner.

If the east line through the center is taken as the X axis and the north line is taken as the Y axis, $I_x = 1{,}000$ and $I_y = 1{,}440$. The moment about the X axis is 10,800 inch-pounds. This moment causes a bending stress of 54 pounds per square inch at the north and south edges. The moment about the Y axis is 14,400 inch-pounds. This moment causes a bending stress of 60 pounds per square inch at the east and west edges. The direct compression is 30 pounds per square inch. The total stresses at the corners are; 144 pounds compression at the northeast corner, 24 pounds compression at the northwest corner, 84 pounds tension at the southwest corner, and 36 pounds compression at the southeast corner.

Problems

1. Solve the example above if the load is placed 3 inches from the east edge and 3 inches from the north edge.

Ans. 111 pounds compression at the northeast corner.

2. In Problem 1, what is the average of the stress at all four corners?

3. A 10-inch by 6-inch post stands vertical with its 10-inch faces running north and south. A load of 2,400 pounds is placed on the top 2 inches from the east face and 2 inches from the north face. A horizontal force

of 75 pounds toward the west is applied to the east face 24 inches above the bottom, and a horizontal force of 80 pounds toward the south is applied to the north face 20 inches above the bottom. Find the unit stress at each corner at the bottom.

Ans. At the northeast corner, compression = 40 + 72 + 40 − 16 − 30 = 106 pounds per square inch.

4. Find the average of the stress at all four corners.
5. From the results for the example above, find the location of the points on the south and west edges at which the stress is zero.

Ans. 8.4 inches from the southwest corner on the south edge, and 7⅞ inches on the west edge.

6. In the example, find the equation of the line of zero stress by means of Equation (1).

$$\textit{Ans.}\ 30 + \frac{3{,}600 \times 4x_1}{1{,}440} + \frac{3{,}600 \times 3y_1}{1{,}000} = 0;$$

$$30 + 10x_1 + 10.8y_1 = 0.$$

7. Check the answer of Problem 5 by substitution in the answer of Problem 6.

The maximum eccentricity in any direction without reversing the stress may be found by equating the second member of Equation (1) to zero,

$$\frac{P}{A} + \frac{Pxx_1}{I_y} + \frac{Pyy_1}{I_x} = 0. \tag{2}$$

This may be written

$$\frac{P}{A}\left(1 + \frac{xx_1}{r_y^2} + \frac{yy_1}{r_x^2}\right) = 0, \tag{3}$$

in which r_x and r_y are the radii of gyration with respect to the X axis and the Y axis, respectively.

Example

Find the maximum eccentricity of the load in the case of a rectangular block of breadth b and thickness d without reversing the stress at the corners.

$$x_1 = \frac{b}{2};\ y_1 = \frac{d}{2};\ r_x^2 = \frac{d^2}{12};\ r_y^2 = \frac{b^2}{12}.$$

The maximum bending stress of sign opposite to the direct stress will be at the corner in the third quadrant if the load is in the first quadrant, and both the second and the third terms of (2) will be negative. Equation (2) becomes

$$1 - \frac{6x}{b} - \frac{6y}{d} = 0. \tag{9}$$

This is the equation of a straight line, the intercepts of which are

$$x = \frac{b}{6},\ y = \frac{d}{6}.$$

CHAPTER XIII

COLUMNS

132. Definition.—In the discussion of eccentric loading in the preceding chapter, no account was taken of the deflection of the body, and of the effect of this deflection upon the eccentricity and the bending stress. Eccentric tension produces a deflection which reduces the eccentricity, as is shown in Fig. 158, III. Eccentric compression, on the other hand, produces a deflection which increases the eccentricity. A yard stick may be placed with one end on the floor and pushed down by the hand at the other end until the middle is bent several inches from the straight line. The original eccentricity, of possibly 0.01 inch, is increased several hundred times, and the bending stress may be sufficient to cause rupture at the middle. If the stick is placed with one end on a platform scale, as shown in Fig. 162, it is found that the load which causes a deflection of 2 inches is little, if any, greater than the load which causes a deflection of 1 inch. While the resisting moment has been doubled, the external moment arm has likewise been nearly doubled. The applied force, therefore, changes very little.

FIG. 162.—A long column.

A compression member whose length is several times as great as its smallest transverse dimension is called a *column* or *strut.* Long vertical compression members of buildings and the posts of bridges are usually called columns. The compression members of roof trusses, and the vertical compression members of airplanes are called struts. The top chord of a bridge usually acts as a column. The connecting rod of an engine is a column during the forward stroke.

When a column is vertical, the only bending moment is that which is due to the eccentricity of the load and to the deflection. When a column is horizontal or inclined, its own weight applied as in a beam becomes an appreciable factor. The rafters which support a roof act as columns and inclined beams.

Illustration

A 1-inch round rod of cold-rolled steel, 36 inches long, was tested as a strut. The ends of the rod were attached to half cylinders of hardened steel which could roll on hardened steel plates. For one test, the axis of each half cylinder was 0.002 inch from the center of the end of the strut. When the load was 8,000 pounds, the deflection of the strut at the middle was 0.0064 inch, and when the load was 10,000 pounds the deflection at the middle was 0.0204 inch. The half cylinders were then moved until the eccentricity was very small. When the load was 9,500 pounds, the deflection was only 0.0002 inch. When the load was raised to 10,500 pounds, the deflection was 0.0011 inch. At 11,000 pounds, the deflection was 0.0055 inch. At 11,200 pounds it was 0.0168 inch. When the load was 11,250 pounds, the deflection at the middle was 0.1739 inch. The load could not be raised above 11,250 pounds. All these deflections were toward the left. With the load of 11,250 pounds on the testing machine and the deflection 0.1739 inch toward the left, a slight horizontal pressure toward the right, applied to the middle of the strut with one finger, bent the strut past the position of zero deflection. It then moved over to the right till the deflection toward the right was 0.1630 inch, while the load on the testing machine still read 11,250 pounds.

Problems

1. In the illustration above, what was the unit stress at the right (concave) surface of the strut when the load was 8,000 pounds?
   Ans. $S = 10{,}186 + 685 = 10{,}871$ pounds per square inch.
2. What was the unit stress at the concave surface of the strut described in the illustration when the eccentricity was 0.0020 inch and the load was 10,000 pounds?
   Ans. $S = 12{,}732 + 2{,}284 = 15{,}016$ pounds per square inch.
3. In the illustration above, what was the stress at the concave surface when the load was 11,250 pounds and the deflection was 0.1739 inch?
   Ans. $S = 34{,}251$ pounds per square inch.

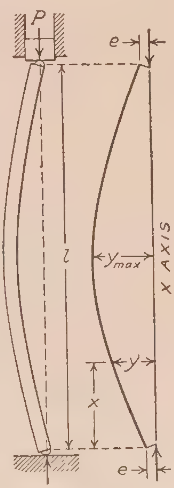


FIG. 163.

133. Column Theory.—Figure 163 shows a vertical column with ends free to turn without friction about a horizontal axis perpendicular to the plane of the paper. The left figure shows the actual column with the deflection somewhat exaggerated. The right figure represents the central axis of the column and the axes of coördinates with all horizontal distances magnified. In order that Formula XV may apply without change of letters, the X axis is taken vertical (parallel to the length of the column) and the Y axis is horizontal and positive toward the left.

This arrangement brings the Y axis 90 degrees from the X axis in the counter-clockwise direction, which is the usual relation. If the figure were rotated 90 degrees in the clockwise direction, the axes would be the same as in common use. In this figure, the X axis is on the line of the applied forces,* and the origin is at the lower end of the column. The eccentricity, which is the distance of the center of gravity of the sections at the end from the line of the load, is regarded as positive. At a section at a distance x from the origin, the moment arm is y and the moment is Py. This moment turns the *lower* end counter-clockwise about the section, and, therefore, is negative. If the load were on the left of the axis of the strut, the eccentricity would be negative (toward the right) and the deflection would also be negative. The moment would then be positive. In either case, $M = -Py$. In Fig. 163, the center of curvature is on the right or negative side of the column. If the eccentricity were in the opposite direction, the center of curvature would be on the opposite side. By this definition, then, the moment is negative when y is positive, and positive when y is negative.

The differential equation for the bent column is

$$EI\frac{d^2y}{dx^2} = -Py. \tag{1}$$

This form of equation is solved by multiplying both sides by dy

$$EI\frac{d^2y}{dx^2}dy = -Py\,dy. \tag{2}$$

$$\frac{d^2y}{dx^2}dy = \frac{dy}{dx}\frac{d^2y}{dx} \tag{3}$$

in which $\frac{d^2y}{dx}$ is the differential of $\frac{dy}{dx}$.

$$EI\frac{dy}{dx}\frac{d^2y}{dx} = -Py\,dy. \tag{4}$$

The integral of Equation (4) is

$$\frac{EI}{2}\left(\frac{dy}{dx}\right)^2 = -\frac{Py^2}{2} + C_1. \tag{5}$$

*In the previous editions of this book, the X axis was the central axis of the column before it was bent. This arrangement made the deflection zero at the ends and the moment arm $y + e$. The present arrangement makes $y = e$ at the ends and makes the moment arm equal to y.

At the middle, where $y = y_{\max}$, $\dfrac{dy}{dx} = 0$.

$$C_1 = \frac{Py^2_{\max}}{2};$$

$$EI\left(\frac{dy}{dx}\right)^2 = P(y^2_{\max} - y^2). \tag{6}$$

$$\frac{dy}{\sqrt{y^2_{\max} - y^2}} = \sqrt{\frac{P}{EI}}\,dx;$$

$$\sin^{-1}\frac{y}{y_{\max}} = \sqrt{\frac{P}{EI}}\,x + C_2. \tag{7}$$

When $x = \dfrac{l}{2}$; $y = y_{\max}$;

$$\sin^{-1}1 = \frac{\pi}{2} = \sqrt{\frac{P}{EI}}\,\frac{l}{2} + C_2; \tag{8}$$

$$\sin^{-1}\frac{y}{y_{\max}} = \frac{\pi}{2} - \sqrt{\frac{P}{EI}}\left(\frac{l}{2} - x\right). \tag{9}$$

$$y = y_{\max}\sin\left[\frac{\pi}{2} - \sqrt{\frac{P}{EI}}\left(\frac{l}{2} - x\right)\right] = y_{\max}\cos\sqrt{\frac{P}{EI}}\left(\frac{l}{2} - x\right). \tag{10}$$

When $x = 0$, $y = e$ and

$$e = y_{\max}\cos\sqrt{\frac{P}{EI}}\,\frac{l}{2}; \tag{11}$$

$$y_{\max} = e\sec\sqrt{\frac{P}{EI}}\,\frac{l}{2} = e\sec\sqrt{\frac{Pl^2}{4EI}}. \quad \text{Formula XXV.*}$$

$$y = e\sec\sqrt{\frac{P}{EI}}\,\frac{l}{2}\cos\sqrt{\frac{P}{EI}}\left(\frac{l}{2} - x\right). \tag{12}$$

Example

A 2-inch by 2-inch timber strut, 5 feet long, is tested as a column with round ends. What is the deflection at the middle under a load of 3,200 pounds, if $E = 1{,}500{,}000$ pounds per square inch and the eccentricity is 0.100 inch?

$$\frac{Pl^2}{4EI} = \frac{3{,}200 \times 60 \times 60 \times 3}{4 \times 4 \times 1{,}500{,}000} = 1.44,$$

$$\sqrt{\frac{Pl^2}{4EI}} = 1.2.$$

$y_{\max} = 0.100 \sec 1.2$ radians $= 0.100 \sec 68° 45' = 0.276$ inch. Deflection $= y_{\max} - e = 0.276 - 0.100 = 0.176$ inch.

* $\sqrt{\dfrac{Pl^2}{4EI}}$ is an angle in radians, the secant (or cosine) of which is a numerical quantity involved in the solution of these column problems. *It does not refer to any angle on the figure.*

Problems

1. In the example above what is the bending moment at the middle, and what is the maximum unit stress?

Ans. $M = 883.2$ inch-pounds; $S = 800 + 662 = 1{,}462$ lb./in.²

2. In the example above, if the load is increased to 3,872 pounds what is the deflection? *Ans.* $y_{max} = 0.403$ inch. $S = 968 + 1{,}170 = 2{,}138$ lb./in.²

3. The 1-inch cold-rolled steel rod described in the illustration of the preceding article was deflected 0.0100 inch at the middle when the load was 5,000 pounds and the eccentricity was 0.0100 inch. The modulus of elasticity was 30,340,000 pounds per square inch. Compute the theoretical deflection and compare with the experimental result.

4. A round-end column, 10 feet long, is deflected 0.050 inch at the middle under a load of 1,000 pounds on the end, and is deflected 0.400 inch under a load of 4,000 pounds. Find the eccentricity by trial and error with a secant table. Then calculate EI.

5. A 2-inch round steel rod, 10 feet long, is used as a column with ends free to turn. Find the deflection at the middle and the maximum fiber stress on the concave side when the load is 8,000 pounds and the eccentricity is 0.1 inch, if E is 30,000,000 pounds per square inch.

Ans. $y_{max} = 0.1 \sec 63° 21' = 0.2230$ inch.
Maximum $S_c = 4{,}814$ pounds per square inch.

6. A column with ends free to turn is made of a 2-inch round steel rod for which E is 30,000,000 pounds per square inch. The length is 5 feet. Find the deflection at the middle and the maximum unit stress for loads of 20,000 pounds, 30,000 pounds, 50,000 pounds, 60,000 pounds, and 70,000 pounds for eccentricities of 0.01 inch and 0.1 inch.

Ans. Load		20,000,	30,000,	50,000,	60,000,	70,000.
Unit stress	For $e = 0.01$,	6,763	10,340	19,295	32,483,	infinite.
	For $e = 0.1$,	10,337	17,500	49,800	154,000,	infinite.

The answers for Problem 6 show that the unit stress for a load of 50,000 pounds with an eccentricity of 0.01 inch is only slightly greater than the unit stress for a load of 30,000 pounds with an eccentricity of 0.1 inch. For a load of 50,000 pounds with an eccentricity of 0.1 inch, the calculated stress is 49,800 pounds per square inch. Ordinary steel would fail under this stress. For a load of 64,750 pounds, the angle $\sqrt{\frac{Pl^2}{4EI}}$ is equal to $\frac{\pi}{2}$ radians. Since the secant of this angle is infinite, a column of these dimensions with a modulus of elasticity of 30,000,000 will fail under this load, no matter how great the ultimate strength of the material. Usually, nothing is gained by using alloy steels of high ultimate strength for columns which are relatively long, since the *critical* load on a column of this kind depends upon

the modulus of elasticity and not upon the ultimate strength of the material.

The formulas of this article were derived under the assumption that E is constant. They are not valid, therefore, beyond the proportional elastic limit. While tests have shown that a short column made of one piece will support a load considerably above the yield point of the material, tests of built up columns show that these fail when the yield point is reached. The yield point of structural steel is a little above the proportional elastic limit. It is best, therefore, to base the factor of safety upon this limit.

Within the elastic limit, these formulas are theoretically and experimentally correct. When the dimensions of the column, the modulus of elasticity of the material, and the *eccentricity* are known, Formula XXV gives the correct moment arm at the section of maximum deflection, and Formula XXIV gives the maximum unit stress.

134. Application of the Secant Formula.—Formulas XXV and XXIV together give the unit stress in a given round-end column under a known load. However, when it is necessary to design or select a column to carry a given load, these formulas are not convenient, since neither the total load nor the unit stress is expressed explicitly. A problem of this kind must be solved by the method of trial and error.

When a number of columns are to be designed, it is a great saving of time to represent the formulas by means of a table or a curve. Before doing this, it is desirable to modify the equations. From Formula XXV,

$$\text{Maximum moment} = eP \sec \sqrt{\frac{Pl^2}{4EI}}. \tag{1}$$

$$\text{Maximum unit stress} = S_u = \frac{P}{A} + \frac{ePc}{I} \sec \sqrt{\frac{Pl^2}{4EI}}; \tag{2}$$

$$S_u = \frac{P}{A}\left(1 + \frac{ec}{r^2} \sec \sqrt{\frac{P}{AE}} \frac{l}{2r}\right), \tag{3}$$

in which r is the radius of gyration of the column. The ratio of the length of the column to its least radius of gyration is $\frac{l}{r}$. This is called the *slenderness ratio* of the column.

To determine the relation of $\frac{P}{A}$ to $\frac{l}{r}$ when the unit stress at the concave surface is the ultimate strength of the material, Equation (3) may be written,

$$\frac{ec}{r^2} \sec \sqrt{\frac{P}{AE}} \frac{l}{2r} = \frac{S_u}{\frac{P}{A}} - 1. \tag{4}$$

It is difficult to solve for $\frac{P}{A}$ in terms of the slenderness ratio, but it is easy to solve for the slenderness ratio in terms of $\frac{P}{A}$.

Table XII gives most of the calculation for structural steel for which the modulus of elasticity is 29,000,000 pounds per square inch and the yield point is 36,000 pounds per square inch, when the eccentricity is such that $\frac{ec}{r^2} = 0.2$.

TABLE XII.—ULTIMATE UNIT LOAD ON A COLUMN WITH ROUND ENDS

$E = 29{,}000{,}000$ and $S_u = 36{,}000$ pounds per square inch; $\frac{ec}{r^2} = 0.2$.

$\frac{P}{A}$	$0.2 \sec \sqrt{\frac{P}{AE}} \frac{l}{2r}$	$\sec \sqrt{\frac{P}{AE}} \frac{l}{2r}$	$\sqrt{\frac{P}{AE}} \frac{l}{2r}$ Degrees	$\sqrt{\frac{P}{AE}} \frac{l}{2r}$ Radians	$\frac{l}{r}$
1,000	35.0	175.	89° 40′	1.565	533.
3,000	11.0	55.	88° 58′	1.553	305.
6,000	5.0	25.	87° 42′	1.531	213.
10,000	2.6	13.	85° 35′	1.494	161.
15,000	1.4	7.	81° 47′	1.427	126.
18,000	1.0000	5.000	78° 28′	1.370	110.
20,000	0.8000	4.000	75° 31′	1.318	100.
22,000	0.6363	3.182	71° 41′	1.251	90.8
24,000	0.5000	2.500	66° 25′	1.159	80.6
25,000	0.4400	2.200	62° 58′	1.099	76.6
26,000	0.3846	1.923	58° 40′	1.024	68.4
27,000	0.3333	1.667	53° 08′	0.927	60.8
28,000	0.2857	1.429	45° 34′	0.795	51.2
29,000	0.2414	1.207	34° 03′	0.594	37.6
30,000	0.2000	1.000	0° 0′	0.000	0.0

To find the value of $\frac{l}{r}$ which makes the maximum unit stress 36,000 pounds per square inch when the unit load is 15,000 pounds per square inch, with $\frac{ec}{r^2} = 0.2$,

$$\frac{36{,}000}{15{,}000} - 1 = 1.4 = 0.2 \sec\left(\sqrt{\frac{15{,}000}{29{,}000{,}000}}\,\frac{l}{2r}\right). \tag{1}$$

$$\sec\left(\sqrt{\frac{15}{29{,}000}}\,\frac{l}{2r}\right) = 7. \tag{2}$$

$$\sqrt{\frac{15}{29{,}000}}\,\frac{l}{2r} = 81^\circ\ 47' = 1.427 \text{ radians}. \tag{3}$$

$$\frac{l}{r} = 2 \times 1.427\sqrt{\frac{29{,}000}{15}} = 126. \tag{4}$$

Curve I of Fig. 164 is drawn from the data of Table XII. Curve II of this figure is for the case when the eccentricity is zero, and is called Euler's curve.

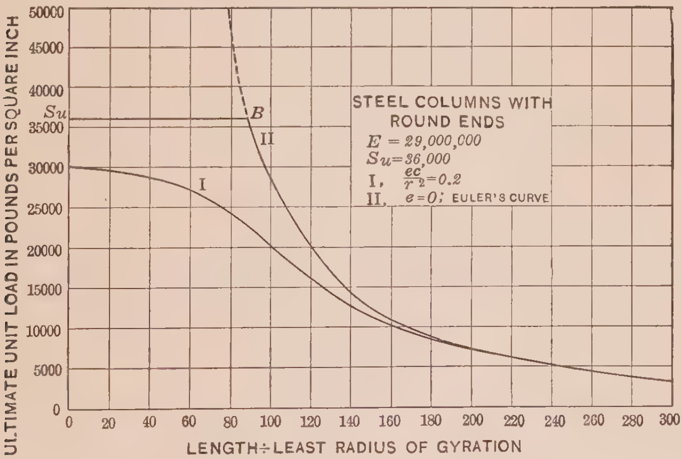


FIG. 164.—Ultimate unit load on steel column.

The two curves approach each other for large values of $\frac{l}{r}$ where the amount of eccentricity makes little difference. For small values of $\frac{l}{r}$ the eccentricity makes a great difference in maximum unit stress.

For values of $\frac{ec}{r^2}$ less than 0.2, other curves could be drawn between curves I and II, and below the horizontal line at 36,000 pounds per square inch.

Example

By means of Curve I of Fig. 164 select the I-beam, 10 feet in length, to be used as a column with round ends to carry a load of 40,000 pounds with a factor of safety of 2.5.

The ultimate load with a factor of safety of 1 is 40,000 × 2.5 = 100,000 pounds. Since the unit load with this eccentricity can not be greater than 30,000 pounds per square inch, the area must be greater than 100,000 ÷ 30,000, which is 3.33 square inches. Since an I-beam of this small section has a very small radius of gyration, the slenderness ratio for such a beam 10 feet in length must be relatively large and the unit load must, therefore, be relatively small. If the slenderness ratio is 120, for instance, the unit load is little over 15,000 pounds per square inch, and the area must be over 6 square inches. A 9-inch, 21-pound I-beam has an area of 6.31 square inches and a least radius of gyration of 0.90 inch. The slenderness ratio is 133 and the unit load, from the curve, is 13,750 pounds per square inch.

$$13{,}750 \times 6.31 = 86{,}700 \text{ pounds.}$$

For a 10-inch, 25-pound I-beam, the area is 7.37 square inches and the least radius of gyration is 0.97 inch. The slenderness ratio is 123, and the unit load is 15,500 pounds.

$$15{,}500 \times 7.37 = 114{,}000 \text{ pounds.}$$

The first of these beams is too small, while the second is larger than necessary. A 9-inch, 25-pound I-beam would come nearer the required figure, but since it is as heavy and expensive as the 10-inch beam, the latter would be chosen.

135. Euler's Formula.—Since the secant of 90 degrees is infinite, Formula XXV shows that any column will deflect without limit, and finally fail, if the unit load and the slenderness ratio are such that

$$\sqrt{\frac{Pl^2}{4EI}} = \frac{\pi}{2}. \tag{1}$$

From Equation (1)

$$P = \frac{\pi^2 EI}{l^2}, \tag{2}$$

which is the common form of Euler's formula. The unit load is

$$\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}. \qquad \text{Formula XXVI.}$$

Formula XXVI is a second form of Euler's formula. In this form, the slenderness ratio $\frac{l}{r}$ and the unit load $\frac{P}{A}$ are the two variables.

Euler's formula contains the moment of inertia, but does not include the distance to the outer fibers. When the eccentricity is negligible, and the slenderness ratio is relatively large, the ultimate load does not depend upon the form of the column, except in so far as the form changes the moment of inertia.

Curve II of Fig. 164 is Euler's curve for a modulus of elasticity of 29,000,000 pounds per square inch. As a mathematical curve, it is of infinite length. As an engineering curve, it must not be used above the point B at which the unit load is the elastic limit of the material.

It will be shown later that it is best not to use Euler's formula for values of the unit load above one-third the elastic limit.

Example

Find the total load with a factor of safety of 2 on a round steel rod 2 inches in diameter, if the elastic limit is 30,000 pounds per square inch and E is 29,000,000 pounds per square inch, for lengths of 40, 60, 80, and 100 inches.

As these lengths are all multiples of 20 inches, begin with this length and find the others by dividing by the square of the ratio. The radius of gyration of a solid circular area being one-half the radius, $r = \frac{1}{2}$ inch. For $l =$ 20 inches,

$$\frac{P}{A} = \frac{9.87 \times 29{,}000{,}000}{1{,}600} = 178{,}894 \text{ pounds per square inch}$$

Length, inches	Unit load, lb./in.^{2}	Total safe load, pounds
40	44,723	
60	19,877	31,223
80	11,181	17,563
100	7,156	11,240

For the 40-inch length $\frac{P}{A}$, as calculated by Euler's Formula, is 44,700 pounds per square inch, which is above the elastic limit, and therefore cannot be used. For the 60-inch length the unit load is less than 20,000 pounds per square inch. This is below the elastic limit, and may be divided by the factor of safety and multiplied by the area of the section to get the total safe load. Euler's formula should not be used for this load unless it is certain that the eccentricity is negligible. For the greater lengths the eccentricity makes little difference, as may be seen from the curves of Fig. 164.

Problems

1. In the example, find $\frac{P}{A}$ for values of $\frac{l}{r}$ from 100 to 300 at intervals of 20. Plot the curve and compare with Fig. 164.

2. The ultimate load on the 1-inch cold-rolled steel rod, 36 inches long, which was described in the illustration of Article 132, was 11,250 pounds. At this load the strut continued to bend with no increase of resistance. It fulfilled, therefore the conditions of Euler's formula. Calculate E.
3. A spruce strut tested at the Bureau of Standards was 1.75 inches square and 6 ft. 3.75 inches long. The ultimate load was 3,020 pounds. Find E by Euler's formula.
4. The compression readings for the strut of Problem 3 were taken for a gage length of 30 inches. When the load changed from 305 pounds to 2,593 pounds, the average compression in the gage length was 0.0098 inch. When the load changed from 305 to 2,745 pounds, the compression in the gage length was 0.0107 inch. When the load changed from 305 to 2,898 pounds, the compression in the gage length was 0.0115 inch. Find E from each of these tests.
5. A yard stick, with the ends slightly rounded, was placed vertical with the lower end on a platform scale and a load was applied to the upper end (Fig. 162). The load and deflection were measured.

Load in pounds	Deflection at the middle, in inches
5.00	0.03
6.00	0.20
6.40	0.25
6.48	1.00 (Load dropped to 6.28)
6.28	2.50

Calculate EI from the last two readings by Euler's formula. *Ans.* 851, 825.
6. The yard stick of Problem 5, supported as a beam at points 34 inches apart, was deflected $^{31}/_{32}$ inch at the middle by a load of 1 pound at the middle. Find EI and compare the result with Problem 1.
7. The yard stick above mentioned was 1.06 inches wide and 0.18 inch thick. Find E and I.

136. Classification of Columns.—Columns may be divided, according to the nature of the ends, into the following classes:

I. Both ends free to turn about horizontal axes but not free to move laterally, Figs. 163 and 165, I.

II. One end fixed and the other end free to turn and free to move laterally, Fig. 165, II.

III. Both ends fixed so that the tangents at the ends do not change, Fig. 165, III.

IV. One end fixed and the other end free to turn about one or more horizontal axes, but not free to move laterally, Fig. 165, IV.

Class I only has been considered in the preceding articles. If L is the entire length of the column, and l is the length of the cosine (sine curve if the origin is shifted) curve of Fig. 163, $L = l$ for Class I.

For Class II, the entire length of the column, from the fixed point B to the top, corresponds with one-half of the cosine curve. Hence, in Formulas XXV and XXVI, $l = 2L$.

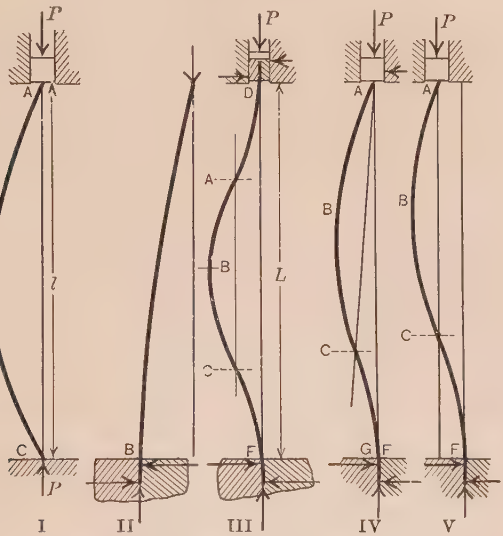


FIG. 165.—Types of ideal columns.

For Class III, $\frac{dy}{dx}$ is zero at each end and at the middle. The middle half ABC corresponds with the cosine curve of Class I. This portion of the cosine curve is represented by l in the formulas. If L is the entire length DF, then $l = \frac{L}{2}$. A column with both ends rigidly fixed will carry as great a load as a column half as long with ends free to turn.

In Fig. 165, III, the points A and C are points of inflection (or counter-flexure) at which the moment and the curvature change signs. The portion AD is equal to one-half of the cosine curve ABC. If revolved 180 degrees in the plane of the paper about the point A, the curve AD will coincide with the curve AB. The moment is zero at A and C.

A column of Class IV is fixed at one end and free to turn at the other but not free to move laterally. The point of counter-flexure is at C. (Fig. 165, IV.) Since the column is free to turn, there is no moment at the top A. Since the moment is zero at the point of counter-flexure C, and there are no transverse forces between A and C, the resultant force from A must pass

through C. The force at A must, therefore, have a horizontal component. The resultant of the horizontal forces at the bottom is equal and opposite to the horizontal component at the top. The portion ABC of Fig. 165, IV, forms a cosine curve with the X axis parallel to CA. The lower portion CF forms part of the cosine curve as far as the plane of the body which holds it. Below that plane it is straight. If this portion continued to curve until it became parallel to AC, it would form a complete half of the cosine curve and its length would be equal to AB or BC. Since the portion is vertical at the fixed end, its length is less than one-half of AB, and less than one-third of the entire length of the column. The solution of the differential equation shows that AC is nearly $0.7L$. For practical purposes, $l = 0.7L$ and $l^2 = 0.5L^2$ nearly.

It is sometimes stated that l is equal to two-thirds L in a column which is fixed at one end and free to turn at the other. This can only be true under the impractical conditions of Fig. 165, V. In this figure, the top of the column is displaced laterally toward the left. If this displacement is such that the point B is as far from the line AC as the top A is from the vertical line through the fixed end F, then the line AC from the end to the point of counter-flexure becomes vertical. In this position, AC is two-thirds of the total length L; there is no horizontal component of the force at the top; and the vertical force is greater than in Fig. 165, IV. The position is unstable. Under a slight vibration the column will deflect to the right of the vertical line through F at the lower end, and the ultimate load will be greatly reduced.

Problems

1. A thin yard stick is clamped vertically in a vise at 4 inches from the lower end. When a load of 2 pounds is placed on the top, the stick deflects with gradually *increasing* speed and, unless supported or the load removed, finally breaks. Find EI. *Ans.* $EI = 830$.
2. A yard stick, with ends rounded, was supported and loaded as in Fig. 165, I, and was deflected a large amount by a load of 6.1 pounds on the top. Find EI by Euler's formula.
3. The yard stick of Problem 2 was clamped 4 inches from one end and the load was applied as in Fig. 165, IV. A deflection of 1.5 inches was caused by a load of 15.42 pounds. Find EI by Euler's formula.
4. The load in Problem 3 was displaced 1 inch south of the vertical line through the bottom. The vertical component of the load when the maximum deflection was 2 inches south was 17.12 pounds. Find EI from this experiment.

137. End Conditions in Actual Columns.—The classification of columns in the preceding article represents ideal conditions, which are only approximated in practice. The columns in actual use are:

Round-end columns, which end with spherical or cylindrical surfaces. They sometimes end with knife-edges, which may be regarded as cylinders of small radii. The round surfaces roll on plane surfaces with practically no friction. Round-end columns are not used in structures and are rarely used in machines.

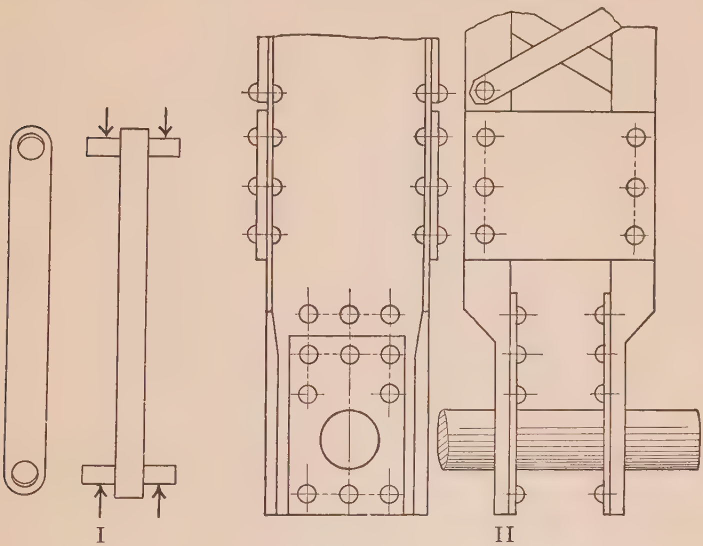


FIG. 166.—Pin-end columns.

Since they meet very closely the conditions of Class I with ends free to turn, they are frequently used in tests to check the accuracy of theory.

A *pin-end* or *hinged-end column* ends with cylindrical surfaces which turn in *cylindrical bearings* (Fig. 166, I). Figure 166, II, shows one end of a pin-connected column made of two channels latticed together. This form of connection is commonly used in bridges. A column which ends with a ball and socket is practically the same as a hinged-end column, except that it is free to turn in any plane instead of in the single plane normal to the axis of the hinge.

Square-end or flat-end columns end with plane surfaces in contact with plane surfaces. The ends must be accurately fitted

to avoid eccentricity. If a beam which rests on a square-end column bends under the load, as shown in Fig. 167, II, the load on the column becomes eccentric. Footings which support columns often settle unevenly and cause large eccentricity.

Pin-end columns are square end in the direction of the axis of the pin.

Fixed-end columns are riveted to the remainder of the structure in buildings and bridges. In a machine, a fixed-end column may be bolted, riveted, or welded to the frame, or may be cast continuous with it. Since the connection can never be absolutely

Fig. 167.—Square-end columns.

Fig. 168.—Deflection of hinged-end column.

rigid, and since the member to which the column is fixed must suffer some deflection, the tangent at the end of the column does not remain exactly "fixed," and the conditions of Class III are never completely satisfied. If the column is very flexible in comparison with the body to which it is fixed, the ideal case is closely approximated and l may be taken as one-half of the total length L. In most practical columns, on the other hand, the assumption that the ends are fixed and that l is equal to one-half L, would introduce a dangerous error.

A column with a pin connection at one end and a square or fixed connection at the other end is called a *pin-and-square* column. This column approximates the conditions of Class IV (Fig. 165, IV) of the preceding article. The yard stick of Problem 2 shows the agreement of experiment with theory. In this experiment, the column was firmly clamped to a 2-inch by 4-inch post and was relatively very flexible $\Big(\frac{L}{r}$ was more than

800). A column of ordinary slenderness fastened to a structure of comparable dimensions would not meet so closely the conditions of the theory, and the experimental and calculated results would not agree so well.

If the pin of a hinged-end column rolled on a smooth plane surface, there would be little friction, and the conditions would be those of the ideal round-end column. Usually the pin turns in a closely fitting seat or bearing, which may introduce considerable friction. If the pin is small, the moment arm of the friction is small and there is little resistance to rotation at the end of the column. If the pin is large, there is considerable resisting-moment, and the column behaves at first approximately as a column with fixed ends. Figure 168 shows diagrammatically three stages of the deflection of a pin-end column.

For a series of tests at the Watertown arsenal in 1909, built I-columns were made of one 10-inch by $\frac{3}{8}$-inch plate and four 4-inch by 3-inch by $\frac{3}{8}$-inch angles. The least radius of gyration was 1.65 inches. The pin-end columns of this series were tested with 3-inch pins which rested in $3\frac{1}{64}$-inch seats. The tests were made on a horizontal compression machine with the axis of each pin vertical and parallel to the 10-inch plate. The results of one test are given in Table XIII.

Tests were made with values of $\frac{l}{r}$ from 25 to 175 inclusive for pin ends as in Table XIII and for square ends. Three columns were tested of each length. The results are given in Table XIV.

Figure 169 is plotted from the averages of Table XIV. Except for the slenderness ratio of 175, there is little difference between the ultimate strength of the hinged-end and that of the square-end columns. One hinged-end column of this length was little over one-half as strong as each of the others, and greatly lowered the average of the three.

Figure 169 also gives Euler's curve for a modulus of 29,000,000. For the slenderness ratios for which Euler's formula applies, this curve is far below the results for pin ends. It is evident, therefore, that a pin connection considerably increases the strength of the column.

Table XV gives the results of a series of tests wrought-iron columns, which were made by the Pencoyd Company.*

* *Transactions of the American Society of Civil Engineers*, 1883, pages 85–122.

Table XIII.—Test of Built I-column at Watertown Arsenal
Area, 13.74 square inches; length, center to center of pins, 24 ft. ½ in.; radius of gyration about axis of pin, 1.65 in.; $\frac{l}{r}$, 175; gage length, 100 inches.

Load, Total	Load, Per square inch	Compression in gage length in inches	Deflection in inches, Horizontal	Deflection in inches, Vertical
13,740	1,000	0	0	0
68,700	5,000	0.0131	0.01	0
137,400	10,000	0.0300	0.01	0
206,100	15,000	0.0468	0.02	0.02
274,800	20,000	0.0642	0	0.02
288,540	21,000	0.0680	0.02	0.02
302,280	22,000	0.0721	0.02	0.03
316,020	23,000	0.0760	0.03	0.04
329,760	24,000	0.0801	0.04	0.04
343,500	25,000	0.0840	0.06	0.04
13,740	1,000	0.0026 set	0.01	0.02
357,240	26,000	0.0874	0.11	0.04
370,980	27,000	0.0940	0.24	0.04
13,740	1,000	0.0052 set	0.04	0.03
371,000	27,010			

Ultimate load. Failed by suddenly springing laterally, after which the resistance was 71,000 pounds.

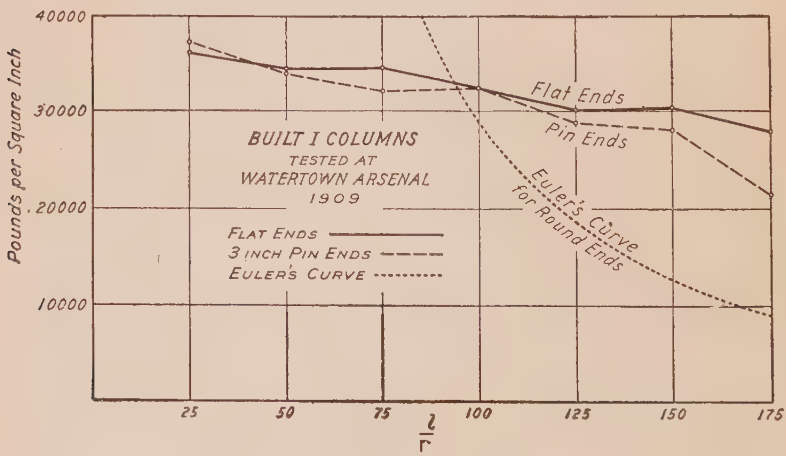


Fig. 169.

TABLE XIV.—COMPARATIVE TESTS OF PIN-END AND SQUARE-END COLUMNS

Slenderness ratio, $\frac{l}{r}$	Ultimate load in pounds per square inch			
	Square ends		Pin ends	
	Separate columns	Average	Separate columns	Average
25	37,450		37,670	
	36,000		37,870	
	35,580	36,343	36,720	37,420
50	34,460		33,800	
	34,560		33,640	
	34,750	34,590	34,080	33,840
75	34,690		32,270	
	34,740		32,000	
	34,420	34,617	32,110	32,160
100	31,670		31,940	
	32,800		31,950	
	32,800	32,423	33,070	32,320
125	29,930		30,000	
	31,300		28,850	
	28,880	30,037	28,740	29,197
150	30,300		27,400	
	30,080		28,310	
	30,520	30,300	29,190	28,320
175	24,730		13,130	
	26,650		27,010	
	26,720	28,033	23,200	21,110

All square-end columns failed by triple flexure. All pin-end columns of slenderness ratio 25 and 50 failed by triple flexure with buckling of the flanges. All pin-end columns with slenderness ratio from 100 to 175, inclusive, failed by sudden springing laterally.

These tests include round-end, hinged-end, flat-end, and fixed-end conditions. In this table, L is the total length of the column, and r is the least radius of gyration. The hinged-ends were ball-and-socket joints, and the round ends were balls on plane surfaces. The columns could bend equally well in any direction, and it was found that failure always

Table XV.—Pencoyd Tests of Wrought-iron Struts

Average Results for Angles and Tees

$\frac{L}{r}$	$\frac{P}{A}$, ultimate unit load in pounds per square inch			
	Round ends	Hinged ends	Flat ends	Fixed ends
20	44,000	46,000	49,000	45,000
40	36,500	40,500	41,000	38,000
60	30,500	36,000	36,500	34,000
80	25,000	31,500	33,500	32,000
100	20,500	28,000	30,250	30,000
120	16,500	24,250	26,500	28,000
140	12,800	20,250	23,250	25,500
160	9,500	16,350	20,500	23,000
180	7,500	12,750	18,000	20,000
200	6,000	10,750	15,250	17,500
220	5,000	8,750	13,000	15,000
240	4,300	7,500	11,500	13,000
260	3,800	6,500	10,250	11,000
280	3,200	5,750	8,750	10,000
300	2,800	5,000	7,350	9,000
320	2,500	4,500	5,750	8,000
340	2,100	4,000	4,650	7,000
360	1,900	3,500	3,900	6,500
380	1,700	3,000	3,350	5,800
400	1,500	2,500	2,950	5,200
420	1,300	2,250	2,500	4,800
440	......	2,100	2,200	4,300
460	......	1,900	2,000	3,800
480	......	1,700	1,900	......

took place in the direction of the least radius of gyration. The figures of this table give some idea of the relative values of the different endings *under the conditions of these experiments.* With a unit load of 25,000 pounds, for instance, $\frac{L}{r}$ is 80 for round ends. For flat ends, this value of the unit load lies between 26,500, for which $\frac{L}{r}$ is 120, and 23,250, for which $\frac{L}{r}$ is 140. Interpolation gives 129 as the value of $\frac{L}{r}$ for flat ends which corresponds with the ultimate load of 25,000 pounds per square inch.

As far *as this experiment goes,* it indicates that the value of l to be used in the calculation of a flat-end column should be about 0.62 of the total length of the column. In a similar way for fixed ends, the value of $\frac{L}{r}$ which corresponds with a unit load of 25,000 pounds per square inch is found to be 144, which makes $l = 0.56\ L$ for this particular case.

In the tests of the hinged-end columns, on account of the lack of vibration, the load was probably greater than would be found in railway bridges subjected to the jar from fast trains. A lubricated hinged-end column, such as the connecting rod of an engine, would probably approximate closely to an ideal round-end column, and L would be nearly equal to l in the formulas.

Problems

1. Using $E = 27{,}000{,}000$ for wrought iron, find the ultimate unit load for slenderness ratios of 160, 200, 300, and 400, and compare with the results for round-end columns in Table XV.

2. Take $\frac{L}{r}$ equal to 60 for round ends in Table XV, and find the equivalent lengths for hinged, flat, and fixed ends. Find the corresponding values of l in terms of L.

Ans. $l = 0.70l;\ l = 0.61L;\ l = 0.63L.$

3. Take $\frac{L}{r} = 100$ for round-end columns and find the corresponding values for hinged, flat, and fixed ends.

Ans. $\frac{L}{r} = 138,\ 160,\ 177.$

If all the values for round ends from 40 to 200 inclusive are taken, and the corresponding values of $\frac{L}{r}$ are determined which give the same unit load for the other conditions, the following ratios are obtained:

	Hinged	Flat	Fixed
Minimum	1.29	1.50	1.25
Maximum	1.45	1.69	1.87
Mean of all	1.37	1.60	1.72

In the case of the fixed ends, only one value was below 1.50.

As far as these figures go, they indicate that a flat-end column 16 feet long, a fixed-end column 17.2 feet long, or a hinged-end column 13.7 feet long, will carry the same total load as a round-end column 10 feet long of the same cross-section.

While riveted and flat-end connections partially fix the ends of the column, the eccentricity is likely to be greater and more uncertain than with hinged ends. The bending stresses, also, which are transmitted from the beams to the columns when the ends are flat or fixed, are equivalent to additional eccentricity. For these reasons, it is safest to regard a fixed end as equivalent to a hinged end and either only a little better than a round end. While formulas are frequently given in handbooks with different constants for the different end connections, it is becoming more and more the practice to use the same formula for all.

CHAPTER XIV

COLUMN FORMULAS USED BY ENGINEERS

138. Straight-line Formulas.—The curves of Fig. 164 show that Euler's formula may be used with little error when $\frac{l}{r}$ is large and that a considerable eccentricity makes little difference. For smaller values of the slenderness ratio, Euler's formula *must not be used*, and a slight difference in the eccentricity makes a relatively large difference in the results of the secant formula. In structures, especially where flat-end or fixed-end columns are used, there is usually considerable uncertainty in regard to the amount of eccentricity. It is, therefore, not worth while to go through the labor of calculating with the secant formulas, except in the cases of relatively large known eccentricity. Engineers make use of simpler approximate formulas. A few years ago Rankine's formula was most used. At present, the *straight-line formulas* have the preference in American practice.

A straight-line formula for the ultimate unit load has the form of

$$\frac{P}{A} = S_u - k\frac{l}{r}, \qquad \text{Formula XXVII.}$$

in which k is a constant, which depends upon the properties of the material. If $\frac{P}{A} = y$ and $\frac{l}{r} = x$, this is recognized as the equation of a straight line with the Y intercept equal to S_u, and with a negative slope equal to k.

If a straight line is drawn in Fig. 164 through the point $(0, S_u)$ and tangent to Euler's curve, this straight line, up to the point of tangency, deviates very little from the secant curve. Except for small values of the slenderness ratio, a small change in the eccentricity would cause the secant curve to pass from one side of the straight line to the other. Such a straight line, then, will give fairly approximate values for the unit loads for the uncertain eccentricities which occur in practice, for all values of $\frac{l}{r}$ to the left of the point of tangency, except very small ones.

Figure 170 shows the method of finding the constant of the straight-line formula graphically. Curve I is Euler's curve for steel for which E is 29,000,000 pounds per square inch. The straight line II is drawn tangent to Euler's curve and passes through point S_u on the Y axis. With S_u equal to 36,000 pounds

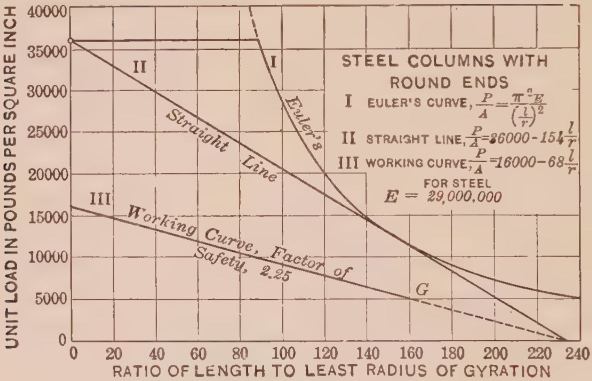


FIG. 170.—Straight line for steel column.

per square inch, the straight line intercepts the X axis at about 233. Since the slope is $36,000 \div 233 = 154$, the straight-line formula for this steel is $\frac{P}{A} = 36,000 - 154\frac{l}{r}$.

This straight-line formula is valid to the point of tangency, or a little beyond that point where it does not differ appreciably from Euler's curve. The straight line may be used for values of $\frac{l}{r}$ up to 160. When the slenderness ratio is more than 160, Euler's equation applies.

Curve III gives the allowable values of $\frac{P}{A}$ with a factor of safety of 2.25. Every ordinate is four-ninths as high as the corresponding ordinate in Curve II. This working straight line ends at G, which is almost directly below the point of tangency. Beyond this point, Euler's curve with each ordinate divided by the factor of safety may be used.

Problem

Plot Euler's curve for timber for which $E = 1,200,000$ pounds per square inch. The ultimate strength of this timber is 4,000 pounds per square inch. Draw the straight line for the ultimate unit loads, and derive a working formula with a factor of safety of 4.

139. Algebraic Derivation of the Straight-line Formulas.—While a straight-line formula may always be derived graphically by drawing Euler's curve and plotting the tangent, the methods of Calculus are convenient and lead to a simple algebraic result. The problem is that of drawing a straight line tangent to a given curve through a given point which is not on the curve. Euler's formula may be written

$$y = \frac{a}{x^2}, \tag{1}$$

in which $y = \frac{P}{A}$, $x = \frac{l}{r}$, and $a = \pi^2 E$.

It is required to draw a tangent to the curve of Equation (1) which shall pass through the point $(0, S_u)$. The slope of this tangent is

$$\frac{dy}{dx} = -\frac{2a}{x_1^3}, \tag{2}$$

in which x_1 is the abscissa of the point of tangency. The equation of the tangent line is

$$y = -\frac{2a}{x_1^3}x + S_u, \tag{3}$$

in which x and y are the coördinates of any point on the line. Since the point of tangency (x_1, y_1) lies in the straight line of Equation (3), these coördinates satisfy the equation of the line; hence

$$y_1 = -\frac{2a}{x_1^2} + S_u. \tag{4}$$

Since the point of tangency is on the curve, these coördinates also satisfy Equation (1); hence

$$y_1 = \frac{a}{x_1^2}. \tag{5}$$

From Equations (4) and (5) the coördinates of the point of tangency are found to be

$$y_1 = \frac{S_u}{3}, \qquad \text{Formula XXVIII.}$$

$$x_1^2 = \frac{3a}{S_u}. \tag{6}$$

The value of x_1 from Equation (6) may be substituted in Equation (3) to get the desired straight-line equation. It is better, however, to use the easily remembered fact of Formula

XXVIII, that the ordinate of the point of contact is one-third the Y intercept of the straight line. When this ordinate is substituted in Euler's formula, the abscissa of the point of contact is found. The coördinates of the point of tangency and of the Y intercept together determine the equation of the straight line.

Example

Derive a straight-line formula for steel for which the ultimate strength is 36,000 pounds per square inch and E is 29,000,000 pounds per square inch.

One-third of the ultimate strength is 12,000 pounds per square inch, which substituted in Euler's formula gives

$$12{,}000 = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2};$$

$$\left(\frac{l}{r}\right)^2 = \frac{9.87 \times 29{,}000{,}000}{12{,}000};$$

$$\frac{l}{r} = 154.$$

$$k = \frac{\frac{2}{3} \times 36{,}000}{154} = 156;$$

$$\frac{P}{A} = 36{,}000 - 156\,\frac{l}{r}.$$

A working formula with a factor of safety of 2.25 is,

$$\frac{P}{A} = 16{,}000 - 69\,\frac{l}{r}.$$

Problems

1. Derive a straight-line formula for timber for which the ultimate strength is 3,000 pounds per square inch and the modulus of elasticity is 1,200,000 pounds per square inch. *Ans.* slope $= 2{,}000 \div 109 = 18.$

$$\frac{P}{A} = 3{,}000 - 18\,\frac{l}{r}.$$

2. Derive a working formula with a factor of safety of 3 for the material of Problem 1.

Ans. $\dfrac{P}{A} = 1{,}000 - 6\,\dfrac{l}{r}.$

3. Derive a straight-line formula with a factor of safety of 5 for cast iron for which S_u is 50,000 pounds per square inch and E is 15,000,000 pounds per square inch.

Ans. $\dfrac{P}{A} = 10{,}000 - 70\,\dfrac{l}{r}.$

140. Connection of Straight-line with Euler's Formula.—A straight-line formula is valid for values of the slenderness ratio which make $\frac{P}{A}$ greater than one-third of S_u (in which S_u is the

first term of the second member of the equation), or which make $k\frac{l}{r}$ less than two-thirds of S_u. Since the tangent leaves Euler's curve gradually at first, the straight-line equation may be used for some little distance beyond the point of tangency with small error, which is on the side of safety.

It seldom happens that a structural column is made with slenderness ratio so large that the straight-line formulas can not be used, but when this does happen, it is necessary to connect with Euler's formula. Straight-line formulas are frequently given with no hint as to the factor of safety or the modulus of elasticity upon which they are based. The formula is frequently derived from a few experiments with relatively short columns, and is likely to have too little slope, since the true curve of ultimate strength is nearly horizontal at first. The formula may be extended for use with long columns and its safety for such columns may be determined by means of the equation of the Euler's curve to which it is tangent. The formula which is now used by the American Railway Engineering Association is

$$\frac{P}{A} = 15{,}000 - 50\frac{l}{r}. \tag{1}$$

At the point of tangency, $\frac{P}{A} = 5{,}000$ and $50\frac{l}{r} = 10{,}000$. The slenderness ratio at the point of tangency is $\frac{10{,}000}{50} = 200$. The equation of Euler's curve to which this straight line is tangent is

$$\frac{P}{A} = \frac{\pi^2 E_w}{\left(\frac{l}{r}\right)^2}, \tag{2}$$

in which E_w is a working modulus of elasticity.

$$E_w = \frac{E}{\text{factor of safety}}.$$

When $\frac{P}{A} = 5{,}000$ and $\frac{l}{r} = 200$, $\pi^2 E_w = 5{,}000 \times 200^2 =$ 200,000,000.

$$E_w = \frac{200{,}000{,}000}{9.87} = 20{,}260{,}000.$$

The factor of safety when Equation (1) is extended to Euler's curve is not quite 1.5. Euler's curve should not be used with this small factor of safety.

Problems

1. The column formula now specified by the building laws of New York and Chicago is

$$\frac{P}{A} = 16{,}000 - 70\frac{l}{r}. \qquad (3)$$

Find the slenderness ratio at which this line is tangent to Euler's curve and find the equation of this curve.

Ans. $\frac{l}{r} = 152.4;\ \pi^2 E_w = 123{,}860{,}000;\ \frac{P}{A} = \frac{123{,}860{,}000}{\left(\frac{l}{r}\right)^2}.$

2. If Equation (3) is extended to long columns by means of Euler's equation, what is the factor of safety? *Ans.* 2.3.

141. Some Straight-line Formulas Which Are Largely Used.—The building laws of New York, Chicago, and many other cities specify for structural steel

$$\frac{P}{A} = 16{,}000 - 70\frac{l}{r}. \qquad \text{Formula XXIX.}$$

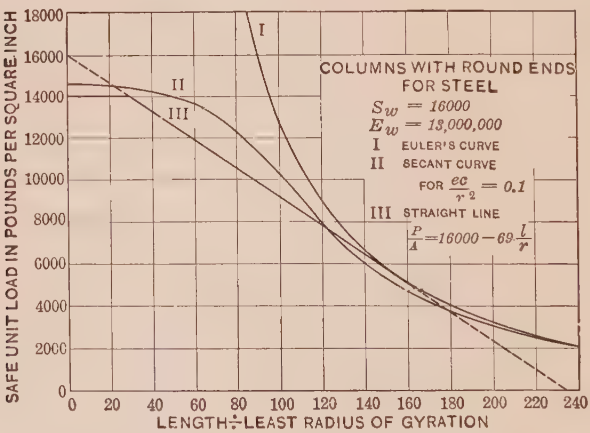


FIG. 171.—Straight line and secant curve for steel.

for values of the slenderness ratio up to 120. Most of these cities specify a maximum unit load of 14,000 pounds per square inch. If the slenderness ratio is less than 2,000/70, so that the calculated unit load comes out greater than 14,000, the maximum of 14,000 is used. The reason for this maximum value below 16,000 pounds per square inch is evident from Fig. 171. Curve II of this figure is the secant curve for material for which the allowable unit stress is 16,000 pounds per square inch and the working modulus of elasticity is 13,000,000 pounds per square

inch, corresponding with a factor of safety of 2.25. The straight line equation from these constants is

$$\frac{P}{A} = 16{,}000 - 69\frac{l}{r} \tag{1}$$

which is practically the same as Formula XXIX. This straight line, III of Fig. 171, is drawn solid from the horizontal line at 14,000 to the point of tangency. The broken portions at the beginning and end of the straight line should not be used. The secant curve is nearly horizontal at first and is only a little above the horizontal straight line. With a slightly larger eccentricity, the secant curve would fall below 14,000 pounds per square inch.

Until recently, Formula XXIX was used by the American Railway Engineering Association. While most specifications limit it to columns for which the slenderness ratio does not exceed 120, it may be used for values of $\frac{l}{r}$ up to 160. Problem 1 of the preceding article shows that it is tangent to Euler's curve at $\frac{l}{r} = 152.4$ and that the continuation as Euler's formula for values of $\frac{l}{r}$ greater than 152 is

$$\frac{P}{A} = \frac{123{,}860{,}000}{\left(\frac{l}{r}\right)^2} \tag{2}$$

Both the straight line and its continuation as Euler's curve have a factor of safety of about 2.25 for round-end columns with a moderate eccentricity. Both formulas are safe for *hinged*, *flat*, or *fixed-end* columns.

The formula now used by the American Railway Engineering Association is

$$\frac{P}{A} = 15{,}000 - 50\left(\frac{l}{r}\right), \tag{3}$$

with a maximum of 12,500 pounds per square inch. The specifications state that the slenderness ratio shall not exceed 100 for main compression members and shall not exceed 120 for wind and sway bracing. This straight line begins lower than that of Formula XXIX and has smaller slope. For small slenderness ratios, it is more conservative than Formula XXIX. For larger ratios, it is less safe. Formula XXIX

may be used up to $\frac{l}{r} = 160$. For even larger values of the slenderness ratio, the error is on the side of safety. It is not advisable to use Equation (3) beyond the limits of the specifications. Formula XXIX may be extended as an Euler's curve. This extension is not recommended for Equation (3). The American Bridge Company uses for structural steel columns

$$\frac{P}{A} = 19{,}000 - 100\frac{l}{r} \tag{4}$$

with a maximum of 13,000 for values of $\frac{l}{r}$ which do not exceed 120. For values of $\frac{l}{r}$ between 120 and 200, the formula is

$$\frac{P}{A} = 13{,}000 - 50\frac{l}{r}. \tag{5}$$

Equation (4) starts higher than Formula XXIX and has greater slope. Equation (5) is parallel to Equation (3) and is always 2,000 pounds per square inch lower. These two equations represent two straight lines which are approximately tangent to Euler's curve at different points.

For cast-iron columns, the building laws of Syracuse specify

$$\frac{P}{A} = 9{,}000 - 40\frac{l}{r}, \tag{6}$$

with a maximum slenderness ratio of 70. This is a conservative formula, and is recommended where it is *necessary* to use cast-iron columns.

For oak and long-leaf yellow pine, the building laws of New York specify

$$\frac{P}{A} = 1{,}200 - 20\frac{l}{D}, \tag{7}$$

in which D is the diameter or least transverse dimension. Since the radius of gyration of a solid circle is one-fourth of the diameter, this corresponds with $1{,}200 - 5\frac{l}{r}$. Problem 2 of Article 139 gives a slope of 6 instead of 5. Since practically all timber columns in buildings have square ends, there is some justification for the slightly smaller slope of Equation (7).

Problems

1. For what value of the slenderness ratio do Formula XXIX and the American Railway Engineering Association formula give the same result for the unit load?

2. Find the value of the unit load at the slenderness ratio of 100 from the American Railway formula, the American Bridge Company formula, and the New York Building Laws.
3. Find the total safe load by Chicago Building Laws for an 8-inch by 8-inch by 1-inch angle section, 10 feet in length, as a compression member. *Ans.* $P = 159{,}220$ pounds.
4. Solve Problem 3 for a length of 3 feet. *Ans.* $P = 210{,}000$ pounds.
5. Find the total safe load on a 10-inch, 25-pound I-beam as a column 10 feet long by Formula XXIX. *Ans.* $\frac{P}{A} = 7{,}340$ pounds per square inch.
6. Solve Problem 5 for a length of 15 feet. *Ans.* $P = 26{,}524$ pounds.
7. Would you solve Problem 5 by the American Railway formula? Why?
8. Find the total safe load by New York Building Laws for a latticed channel column 20 feet in length, which is made of two 10-inch, 20-pound channels placed 6 inches back to back.
9. Solve Problem 8 by the American Bridge Company formula.
10. A plate-and-channel column, 24 feet long, is made of two 8-inch by ¼-inch plates and two 6-inch, 8-pound channels placed 3⅝ inches back to back. Find the total safe load by two suitable formulas.
11. Find the total safe load on an 8-inch by 6-inch by 1-inch angle, 16 feet in length, by the American Bridge Company formula and by some other suitable straight-line formula.
12. Find the safe load on an 8-inch by 12-inch oak post, 15 feet long, by New York Building Laws. *Ans.* 72,000 pounds.
13. A long-leaf yellow pine post, of rectangular section, is 12 feet long and 10 inches wide. What must be its thickness to carry a load of 60,000 pounds? *Ans.* 7.4 inches. Use an 8-inch by 10-inch post.
14. Solve Problem 13 for a load of 100,000 pounds. *Ans.* 10.97 inches.
15. By Formula XXIX, find the diameter of a solid steel cylinder, 10 feet long, which carries a load of 100,000 pounds in compression. *Ans.* 4.06 inches.

142. Rankine's or Gordon's Formula.—While the straight line formulas have recently come into general use among American engineers, on account of the ease of application and the fact that they agree as well with the results of tests and with exact theory as the more complicated expressions, another type of working formula had the preference until a few years ago, and is still the favorite with British engineers. This type is called Gordon's or Rankine's formula. It is an empirical formula, which gives the unit load equal to the ultimate strength for a short block and which approaches Euler's curve for a very long column. The formula is

$$\frac{P}{A} = \frac{S_u}{1 + q\left(\frac{l}{r}\right)^2}, \qquad \text{Formula XXX.}$$

in which S_u is the ultimate unit load in compression on a short block of the material, and q is a coefficient, the value of which may be determined experimentally. The allowable unit load is found by dividing by the factor of safety. This is equivalent to using the allowable compressive stress instead of the ultimate strength as the numerator of the formula.

Example

The Philadelphia Building Laws specify for medium steel columns in buildings

$$\frac{P}{A} = \frac{16{,}250}{1 + \dfrac{l^2}{11{,}000r^2}}. \tag{1}$$

Find the total safe load on a solid 4-inch cylinder, 10 feet in length, as a column.

$$r = 1 \text{ inch}; \frac{l}{r} = 120; \frac{l^2}{11{,}000r^2} = \frac{14{,}400}{11{,}000} = 1.3091.$$

$$\frac{P}{A} = \frac{16{,}250}{2.3091} = 7{,}037 \text{ lb./in.}^2$$

$$P = 7{,}037 \times 12.5664 = 88{,}430 \text{ lb.}$$

Problems

1. Find the total safe load by the Philadelphia formula for an 8-inch by 6-inch by ½-inch angle of medium steel as a column 13 feet in length. *Ans.* 47,500 pounds.
2. Find the total safe load by Philadelphia formula on a 15-inch, 42-pound I-beam as a column 15 feet long. *Ans.* P = 57,530 pounds.
3. Solve Problem 8 of the preceding article by the Philadelphia formula.

The values of the total safe load on columns which are given in Cambria Steel Handbook are calculated from the formula

$$\frac{P}{A} = \frac{12{,}500}{1 + \dfrac{l^2}{36{,}000r^2}} \tag{2}$$

for square ends. The value of q for pin-ends is $\dfrac{1}{18{,}000}$. These are Rankine's constants and are based on a limited number of tests of relatively short columns. Both are too small and err on the side of danger. On the other hand, the allowable unit stress of 12,500 pounds per square inch is more conservative than the figures generally used. The tables, therefore, give good values for all columns except relatively long ones. The

figures for these are safe, but the factor of safety decreases as the length of the columns increases.

Problems

4. Find the total safe load on a 10-inch, 35-pound I-beam as a column 16 feet in length by Equation (2) and compare with the Cambria tables.
5. Find the total safe load on a plate-and-angle column, made of one 8-inch by ½-inch plate and four 4-inch by 3-inch by ½-inch angles, for lengths of 16 feet and 32 feet. Compare with Cambria tables.
6. Solve Problem 5 by some suitable equation from the preceding article.

143. Ritter's Rational Constant for Rankine's Formula.—While the coefficient q was originally derived from a few tests of columns, it may be obtained from the constants of the material of the column. Both experiment and theory show that Euler's formula gives the ultimate load when the eccentricity is very small, the ends are perfectly free to turn or absolutely fixed, and the slenderness ratio is so large that the computed unit load is below the proportional elastic limit of the material. Any equation which is to be valid for all lengths must agree with Euler's curve when the slenderness ratio becomes indefinitely large, and must give the ultimate strength of the material when the slenderness ratio is zero. A straight-line equation for the ultimate strength passes through S_u when $\frac{l}{r}$ is zero, and coincides with Euler's curve at the point of tangency. Beyond the point of tangency, Euler's equation must be used. When $\frac{l}{r} = 0$ in Rankine's formula, the denominator is unity, and $\frac{P}{A} = S_u$. Rankine's formula, therefore, satisfies one condition. To make it satisfy the other condition, the value of q must be so chosen that $\frac{P}{A}$ shall be the same in Rankine's and Euler's formulas for large values of the slenderness ratio.

$$\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} = \frac{S_u}{1 + q\left(\frac{l}{r}\right)^2}. \tag{1}$$

For large values of $\frac{l}{r}$ the second term in the denominator of Ran-

kine's formula is so large relatively that the first term (unity) may be dropped. Then

$$\frac{\pi^2 E}{\left(\frac{l}{r}\right)^2} = \frac{S_u}{q\left(\frac{l}{r}\right)^2} \tag{2}$$

$$q = \frac{S_u}{\pi^2 E} \tag{3}$$

This value of q is *Ritter's rational constant.*

Problems

1. Find the value of q for steel for which the modulus of elasticity is 29,000,000 pounds per square inch, and the ultimate compressive strength is 36,000 pounds per square inch. *Ans.* $q = \frac{1}{7{,}950}$

2. Find the unit load in pounds per square inch for values of the slenderness ratio at intervals of 40 from 40 to 200 if $q = \frac{1}{8{,}000}$ and $S_u = 36{,}000$ pounds per square inch.

Ans.

$\frac{l}{r}$	40	80	120	160	200
$\frac{P}{A}$	30,000	20,000	12,857	8,571	6,000

3. Find the value of $\pi^2 E$ which makes $q = \frac{1}{8{,}000}$ when S_u is 36,000 pounds per square inch. From this value solve Problem 2 by Euler's formula.

4. Extend the solutions of Problems 2 and 3 up to $\frac{l}{r} = 400$.

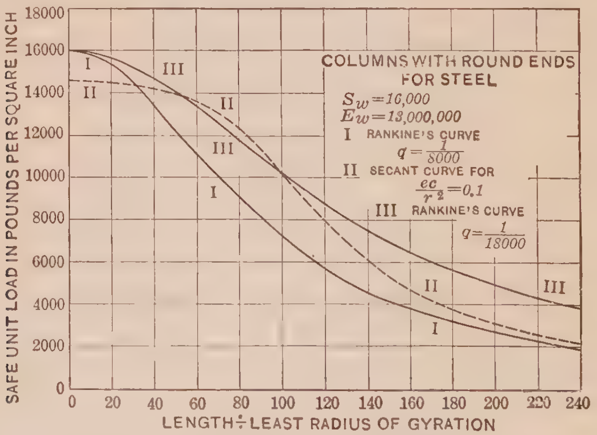


FIG. 172.—Curves for steel columns.

Curve I of Fig. 172 is drawn from Rankine's formula for steel with an allowable unit stress of 16,000 pounds per square inch and a working modulus of elasticity of a little less than

13,000,000. These constants make $q = \frac{1}{8,000}$ and give a factor of safety of about 2.25 for both the compressive stress and the modulus of elasticity. Curve II is drawn from the secant formula with $\frac{ec}{r^2} = 0.1$. For small slenderness ratios, Curve I is above Curve II. For most values of $\frac{l}{r}$ the Rankine's curve with Ritter's constant errs on the side of safety.

Curve III is calculated from $q = \frac{1}{18,000}$, which is Rankine's constant for pin-end columns. For relatively short columns, the results are approximately correct, but for long columns there is a large error on the side of danger.

The constant of the Philadelphia Building Laws is $\frac{1}{11,000}$ for medium steel, which represents a fair compromise between $\frac{1}{8,000}$ and $\frac{1}{18,000}$. Since most columns in buildings have square or fixed ends, and the remainder have hinged ends, which are partly fixed by friction, there is some justification in using a constant which brings the unit load for large slenderness ratios slightly above the results of Euler's equation. The Philadelphia formula,

$$\frac{P}{A} = \frac{16,250}{1 + \frac{l^2}{11,000r^2}}, \tag{4}$$

is recommended, therefore, as a satisfactory equation for structural steel columns of any length.

The Philadelphia Building Laws specify for mild steel

$$\frac{P}{A} = \frac{14,500}{1 + \frac{l^2}{13,500r^2}} \tag{5}$$

This, also, is a good formula for columns of any kind of structural steel. It may be used for all values of the slenderness ratio.

Problems

5. Find $\frac{P}{A}$ by Equation (5) for values of $\frac{l}{r}$ from 20 to 200 at intervals of 20. Compare with the American Bridge Company formulas.
6. Solve Problem 5 of Article 142 by means of the Philadelphia formula for medium steel.

144. Selection of a Column for a Given Load.—The problem of designing or selecting a column of a given length to carry a given load varies with the form of the section. If the sections which are considered are all similar figures, the radius of gyration varies as the first power and the area varies as the second power of any dimension. For a circle of radius a, for instance, $r = \frac{a}{2}$ and $A = \pi a^2$. For a square of side b, $r = \frac{b}{\sqrt{12}}$ and $A = b^2$. A problem of this class may be solved algebraically for the unknown dimension. Euler's equation gives the fourth power of this unknown quantity (since the moment of inertia varies as the fourth power). The required result is obtained by extracting the square root of a square root. A straight-line formula gives a quadratic equation. Rankine's formula gives a quadratic equation in terms of the square of the unknown dimension. Any one of these equations may be easily solved.

Example I

A square steel bar, as a column 10 feet long, carries a load of 12,000 pounds. Find its dimensions by the working Euler's formula, Equation (2) of Article 141.

$$\frac{P}{A} = \frac{123,860,000}{\frac{120^2}{r^2}}. \tag{1}$$

If b is one side of the square, $r^2 = \frac{b^2}{12}$ and $A = b^2$.

$$\frac{12,000}{b^2} = \frac{123,860,000b^2}{120 \times 120 \times 12}; \tag{2}$$

$$b^4 = \frac{120 \times 120 \times 12 \times 12,000}{123,860,000} = 16.741; \tag{3}$$

$$b^2 = 4.09;\ b = 2.02.$$

Example II

A square steel bar as a column 10 feet long, carries a load of 65,000 pounds. Find its dimensions by Formula XXIX.

$$\frac{65,000}{b^2} = 16,000 - \frac{70 \times 120\sqrt{12}}{b}; \tag{4}$$

$$65,000 = 16,000b^2 - 8,400\sqrt{12}b; \tag{5}$$

$$160b^2 - 84\sqrt{12}b - 650 = 0; \tag{6}$$

$$b = 3.12 \text{ inches.}$$

Example III

Solve Example II by the Philadelphia formula for medium steel.

$$\frac{65{,}000}{b^2} = \frac{16{,}250}{1 + \dfrac{120^2 \times 12}{11{,}000b^2}}; \tag{7}$$

$$\frac{b^2}{4} = \frac{11{,}000b^2 + 172{,}800}{11{,}000b^2}; \tag{8}$$

$$110b^4 - 440b^2 - 6{,}912 = 0; \tag{9}$$

$$b^2 = \frac{440 + 1{,}798}{220} = 10.17$$

$$b = 3.19 \text{ inches.}$$

Problems

1. Solve Example I for a solid circular section.
2. Solve Example II for a solid circular section.
3. Solve Example III for a hollow section with the inside diameter one-half the outside diameter.
4. Find the value of the slenderness ratio for each of the examples above.

Since the sections of rolled shapes of different sizes are not similar figures, the selection of a column must be made by the method of trial and error. The steel handbooks make it possible to select a column which is approximately correct. Carnegie gives a table which contains the unit loads for values of the slenderness ratio at intervals of 5 as calculated from the American Bridge Company formula, the American Railway Engineering Association formula, Gordon's formula with Rankine's constant for fixed ends, and the building laws of nine of the principal cities. This handbook also gives the total safe loads for a series of I-beams as columns and for a large number of plate-and-channel and plate-and-angle columns. These are computed by the American Bridge Company formulas. Where these formulas are specified, the tables may be used directly with no calculation. Cambria Steel gives the safe load on a great variety of shapes and built columns. These are calculated by Rankine's formula with a constant of $\frac{1}{36{,}000}$ and a maximum unit load of 12,500. For small slenderness ratios, this formula gives larger columns than most of the other column formulas. For relatively long columns, on the other hand, the reverse is true.

Example IV

Select a plate-and-channel column, 30 feet long, to carry a load of 100,000 pounds by the New York Building Laws.

A 30-foot column made of two 10-inch by ¼-inch plates and two 8-inch, 11.25-pound channels according to Cambria Steel will carry a load of 104,000 pounds. The area of this column is 11.70 square inches and the least radius of gyration is 2.98 inches.

$$\frac{P}{A} = 16{,}000 - \frac{70 \times 360}{2.98} = 7{,}544 \text{ pounds}$$

per square inch. Since this is more than one-third of 16,000, the slenderness ratio does not exceed the limits for the application of this formula.

$$7{,}544 \times 11.70 = 88{,}265 \text{ pounds},$$

which is about 12 per cent. too small. An inspection of the table shows that the column with ⅜-inch plates has an area of 14.20 square inches and a least radius of gyration of 2.97. The unit load for this section and length is 7,515 pounds per square-inch and the total load is 106,710 pounds.

Problems

5. Find the I-beam as a column 14 feet long to carry 50,000 pounds by an American Bridge Company formula.
6. Select a latticed channel column, 28 feet high, to carry a load of 200,000 pounds by the Philadelphia formula for medium steel.
7. Solve Problem 6 by the New York Building Laws.

Solve the next four problems algebraically, if possible.

8. A rectangular steel column is 4 inches wide and 15 feet long. Find the thickness to carry a load of 180,000 pounds by the New York Building Laws.
9. Solve Problem 8 if the load is 480,000 pounds.
10. Solve Problem 8 by the working Euler's formula, Equation (2) of Article 141
11. Solve Problem 8 by Philadelphia Building Laws for medium steel.

145. Concrete Columns.—For concrete columns, it is customary to specify an allowable load in pounds per square inch, and a maximum ratio of length to diameter or minimum breadth. Most cities specify a maximum unit load of 500 pounds per square inch and a maximum $\frac{L}{D}$ of 15. The report of the Joint Committee on Concrete and Reinforced Concrete gives 450 pounds per square inch for 1:2:4 concrete which has an ultimate strength of 2,000 pounds per square inch.

Problems

1. Concrete columns, 14 feet long, are designed to carry a floor load of 400 pounds per square foot. The columns are circular and are spaced 12 feet apart one way and 15 feet apart the other way. What is the diameter of each column?
2. Solve Problem 1 if the columns are 16 feet apart each way.
3. Solve Problem 1 if the columns are 20 feet long.

146. General Conclusions.—The calculation of the strength of columns is not as satisfactory as that of beams. This is due to two reasons: the location of the load and the relative freedom of the ends. In a beam, the location of the load is known with a large relative accuracy. A 1-inch displacement of the load in a horizontal beam 10 feet long produces a very small change of the maximum unit stress; an equal displacement of the load at the end of a block 6 inches square *doubles* the maximum stress if the block is short, and has still greater effect if the block is relatively long. Again, most beams are entirely free to turn at the supports, or are perfectly fixed at one end, so far as the moment is concerned, and entirely free to turn and move at the other end. The results obtained in the calculation of beams are correct inside the elastic limit and are approximately true beyond that limit. If a column is perfectly free to turn at the ends, and if the location of the load is known with the same relative accuracy as in a beam, the unit stress may be calculated with the same relative accuracy in both. There is this apparent difference: in a beam the unit stress varies with the load; in a column it increases more rapidly. Again, a column which is fixed at one end and free to move and turn at the other may be calculated with the same accuracy as a cantilever with one end free, provided the load is located with the same relative accuracy and the end is so well fixed that the *relative* change in moment which is due to change in tangent at the "fixed end" is the same for both. The change in moment which is caused by a change in the tangent at the fixed end is proportional to the rate of change of the cosine of a small angle in the case of a beam, and is proportional to the rate of change of the sine of a small angle in the case of a column. The change in moment is, therefore, much greater in a column than in a beam for the same change of angle. Since the load is greater in a column than in a beam of equal section, the change in moment which is caused by a change in slope at the end is increased still further.

Beams fixed at both ends or fixed at one end and supported at the other are indefinite, because it is not possible to fix the beam perfectly so that it will not turn, or support it so that it will not move. For these reasons the calculation of the unit stresses in relatively stiff beams of these kinds is always open to question. The same is true of columns fixed at both ends, or fixed at one end with a hinge connection at the other.

The error in the case of the fixed column is relatively greater than in a fixed beam, for a change in the slope of the tangent at the ends of the column makes a relatively larger change in the bending moment.

Euler's formula gives the ultimate load which will cause a column with practically no eccentricity to deflect without limit. Unless the slenderness ratio is large the column will fail by crushing before this load is reached. *If the value of* $\frac{P}{A}$, *calculated by Euler's formula is greater than the elastic limit of the material, it must be discarded and the calculation repeated with a formula which fits shorter columns. It is best to limit the use of Euler's formula to slenderness ratios where it gives values of* $\frac{P}{A}$ *which, after division by the factor of safety, are less than one-third of the allowable unit compressive stress of the material.*

For shorter columns draw a straight line through the proportional elastic limit of the material and tangent to Euler's curve and divide by the factor of safety, or construct an Euler's curve by means of a working modulus of elasticity obtained by dividing E by the factor of safety and draw a line tangent thereto through an intercept on the Y axis of which the ordinate is the allowable unit compressive stress.

The effect of eccentricity is taken into account by using a limiting stress for short columns, as in the case of the American railway formula, and by the use of a large factor of safety (well called a factor of ignorance) to take care of any uncertainties in this respect. (The real factor of safety in many columns which are standing is probably much less than figured by the designer.)

Rankine's formula is used by some engineers. With Ritter's constants it is always safe—unnecessarily safe for columns of moderate length. With Rankine's constants it should not be used for long columns.

If the eccentricity of the load were sufficiently well known, the secant formulas of Article 134 are strictly correct for round-end columns, provided the stress is below the elastic limit. A set of curves like those of Fig. 164 may be drawn and employed in the calculations to save labor.

Riveted ends, hinged ends, and square ends fix the ends to a greater or less degree. Riveted ends and flat ends are sometimes considered as perfectly fixed and half the total length is

used as l in the equations. This is dangerous practice. More frequently some factor is used to change the coefficient or the effective length of the column in the calculations. The determination of such a factor is a question of engineering judgment in each design. Most practical specifications, at present, use the same formulas and constants for all kinds of end connections except the unusual one, a column which is fixed at one end and is free to turn and move at the other end. In calculating a column of this kind, the length in the formulas must be twice the length of the column.

147. Failure of a Beam by Buckling of the Compression Flange.—The compression flange of a beam acts as a column and may fail by lateral deflection. In the calculation of this failure, the unit bending stress in the extreme outer fibers is taken as the unit load $\frac{P}{A}$ of the column formulas. Unless the moment is constant in the beam, the unit stress increases from the end to the middle and the compression flange of one half a beam which is supported at the ends is equivalent to a column which is fixed at one end, free at the other, and carries a load which is distributed along its length. The problem is not, therefore, exactly the same as that of an ordinary column.

The American Railway Engineering Association specifications (Second Edition, May, 1923) state that the stress per square inch in the compression flange of an I-beam shall not exceed

$$S_c = 16{,}000 - 150\frac{l}{b}, \tag{1}$$

in which l is the length of the unsupported flange, between lateral connections or knee braces, and b is the flange width. Since the flange may be regarded as a rectangle of breadth b, its lateral radius of gyration is $\frac{b}{\sqrt{12}}$. Equation (1) is, therefore, equivalent to a column formula

$$\frac{P}{A} = 16{,}000 - 43.3\frac{l}{r}. \tag{2}$$

Since the load is not applied at the end but is distributed, there is some justification for the smaller constant of 43 instead of 50 which is used in the American Railway formula for columns.

The American Bridge Company recommends the formula

$$\text{Unit compressive stress} = 19{,}000 - 300\frac{l}{b} \qquad (3)$$

with a maximum of 16,000 pounds per square inch.

Example

Calculate by Equation (1) the maximum distance between lateral supports for a 12-inch, 31.5-pound I-beam if the maximum bending stress is 14,000 pounds per square inch.

$$14{,}000 = 16{,}000 - 150\frac{l}{5};$$
$$30l = 2{,}000;$$
$$l = 67 \text{ inches.}$$

Problems

1. Find the maximum allowable bending stress in an 18-inch, 55-pound I-beam, 15 feet long, with no lateral supports for the compression flange by the American Railway formula.
   Ans. $S_c = 11{,}500$ pounds per square inch.
2. Solve Problem 1 by the American Bridge Company formula.
3. Find the I-beam for a span of 20 feet to carry a distributed load of 6,000 pounds per foot with a maximum unit stress of 15,000 pounds per square inch. How many lateral stiffeners will be needed?

The table in Cambria in the article entitled "Lateral Strength of Beams without Lateral Support" was derived from the Rankine formula for fixed-end columns.

$$\frac{P}{A} = \frac{18{,}000}{1 + \dfrac{l^2}{36{,}000r^2}} = \frac{18{,}000}{1 + \dfrac{l^2}{3{,}000b^2}} \qquad (4)$$

by substituting b^2 for $12r^2$. The numerator is taken at 18,000 pounds per square inch instead of the usual 16,000 pounds per square inch on account of the fact that only a part of the flange is subjected to the maximum unit stress.

Problems

4. Find the total safe load, uniformly distributed, on a 15-inch, 42-pound I-beam, 22 feet long, which is supported at the ends and has no intervening lateral supports.
   Ans. Maximum $S_c = 10{,}181$ pounds per square inch; total load = 18,170 pounds.
5. Solve Problem 4 by Equation (1).

There is a considerable variation in the distance between lateral supports of the compression flange of a beam as given by the

different formulas.* However, any one of these formulas is safe; they differ only in the magnitude of the factor of safety. The dangerous condition is that of the inexperienced engineer *who forgets to use any formula and computes his I-beams for bending stress alone.*

148. Failure of Beam by Buckling of the Web.—It was shown in Article 31 that vertical shear produces compressive stress, which is a maximum at 45 degrees with the vertical, and that this

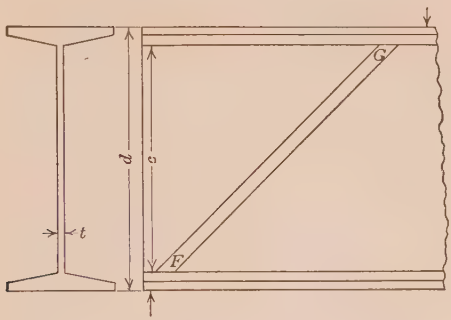


FIG. 173.—Web of I-beam as a column.

maximum compressive stress is equal to the unit vertical or horizontal shearing stress. The web of an I-beam subjected to vertical shear may be regarded as made up of a series of parallel columns, as *FG*, Fig. 173, with fixed ends. The thickness of this column is t, the thickness of the web. The vertical distance between the flanges is c. (This distance c of Fig. 173 is the same as l of the Cambria diagrams.) Since the column *FG* makes an angle of 45 degrees with the vertical, its length is $c\sqrt{2}$. The average vertical shear in the web of an I-beam is found by dividing the total vertical shear by the area td, in which t is the thickness of the web and d is the total depth of the beam. Since the unit compressive stress in the inclined column *FG* is equal to the unit shearing stress in the web,

$$S_c = \frac{V}{td}. \qquad (1)$$

* A full discussion of this subject is to be found in *Bulletin* No. 68 of The Illinois University Engineering Experiment Station, by PROF. HERBERT F. MOORE.

See also paper by R. FLEMING in *Engineering News*, April 6, 1916, and paper by HENRY KERCHER in *Engineering News*, May 4, 1916.

This unit compressive stress is equivalent to $\frac{P}{A}$ of the column formulas. To find the safe value of V it is only necessary to solve for $\frac{P}{A}$ by any column formula, remembering that $r^2 = \frac{t^2}{12}$ for a rectangular section of thickness t. The column FG may be regarded as 1 inch wide and t thick. Cambria Steel uses Rankine's formula with a numerator of 12,000 pounds per square inch and $q = \frac{1}{36{,}000}$.

$$\frac{P}{A} = \frac{12{,}000}{1 + \frac{l^2}{36{,}000r^2}} = \frac{12{,}000}{1 + \frac{c^2}{1{,}500t^2}}. \tag{2}$$

Problems

1. Find the maximum value of the unit shear, the total vertical shear, and the total load uniformly distributed, on a 12-inch 31.5-pound I-beam, by means of the above formula.
   Ans. 7,488 pounds per square inch, 31,450 pounds, and 62,900 pounds.
2. Solve Problem 1 for a 15-inch 42-pound I-beam. Compare results with Cambria under "Maximum Loads of I-beams and Channels Due to Crippling the Web."
3. If the allowable unit stress due to bending is 16,000 pounds per square inch, what is the minimum length for which the full bending stress may be developed by a uniformly distributed load without producing excessive buckling stresses in a 12-inch 31.5-pound I-beam?
4. Solve Problem 3 for a 20-inch 65-pound I-beam for the maximum load and minimum span without crippling the web if the load is concentrated at the middle.
5. Solve Problem 1 by the American Railway column formula.
6. Solve Problem 2 by New York Building Laws.

CHAPTER XV

COMBINED STRESS

149. Resultant of Shearing and Tensile Stress.—Figure 174 represents a block of breadth dx, height dy, and length l, subjected to tensile stresses of intensity s_t perpendicular to the left and right vertical faces, to shearing stresses of intensity s_s parallel to these faces, and to shearing stresses of equal intensity in the

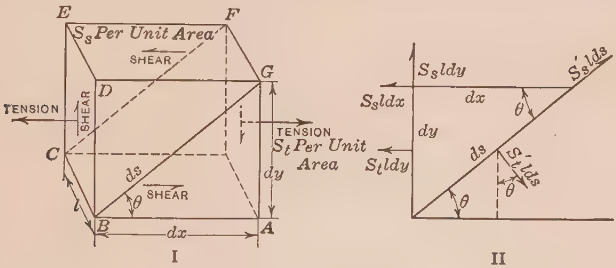


FIG. 174.—Combined shear and tension.

top and bottom faces. The shear on the left face is upward and on the top face toward the left. It is desired to find the unit shearing stress parallel to the diagonal BG or CF and the unit tensile stress normal to the plane $BCFG$. The block may be considered as divided by the plane $BCFG$ into two equal triangular prisms. The prism which lies to the left of this plane will be taken as the free body in equilibrium. The forces which act on this free body are five in number:

Total tension $s_t ldy$, toward the left, applied at center of $BCED$;

Total shear $s_s ldy$, upward, applied at center of $BCED$;

Total shear $s_s ldx$, toward the left, applied at center of $DEFG$;

Total shear on $BCFG$, parallel to BG, applied at center of $BCFG$;

Total tension normal to $BCFG$ at its center.

The unknown unit shearing stress in the plane $BCFG$ will be represented by s_s' and the unknown unit tensile stress by s_t'

The total shear on this plane is then $s_s'lds$, where ds is the length of the diagonal BG. The total tension on the diagonal plane is $s_t'lds$. The five forces which act on the wedge $BCEDFG$ are represented in a single plane in Fig. 174, II.

The magnitude of the unknown shearing stress s_s' may be found by resolving parallel to the line BG. If θ is the angle between $BCFG$ and the horizontal, the resolution parallel to BG, after dividing by l, is

$$s_t dy \cos\theta + s_s dx \cos\theta - s_s dy \sin\theta = s_s' ds. \tag{1}$$

When Equation (1) is divided by ds, and $\dfrac{dx}{ds}$ and $\dfrac{dy}{ds}$ are expressed in terms of the cosine and sine of θ, the result is

$$s_s' = s_t \sin\theta\cos\theta + s_s(\cos^2\theta - \sin^2\theta); \tag{2}$$

$$s_s' = \frac{s_t}{2}\sin 2\theta + s_s \cos 2\theta. \tag{3}$$

The resolution normal to ds gives

$$s_t dy \sin\theta + s_s dx \sin\theta + s_s dy \cos\theta = s_t' ds; \tag{4}$$

$$s_t' = s_t \sin^2\theta + 2s_s \sin\theta\cos\theta; \tag{5}$$

$$s_t' = \frac{s_t}{2}(1 - \cos 2\theta) + s_s \sin 2\theta. \tag{6}$$

These equations apply when the external shearing stresses on the block have the directions of Fig. 174. If the shear is reversed. some of the signs are changed.

Problems

1. A block is subjected to a horizontal tensile stress of 320 pounds per square inch and a horizontal and vertical shearing stress of 200 pounds per square inch, as in Fig. 174. Find the resultant unit shearing stress at 20° with the horizontal.

Ans. $s_s' = 256.0$ pounds per square inch.

2. In Problem 1, find the resultant unit tensile stress across a plane which makes an angle of 20° with the horizontal.

Ans. $s_t' = 166.0$ pounds per square inch.

150. Maximum Resultant Unit Shearing Stress.—The direction which the plane $BCFG$ of Fig. 174 should have in order that the unit shearing stress in it shall be a maximum is found by

differentiating the expression for s_s' of Equation (3) of the preceding article with respect to θ. This is

$$\frac{d}{d\theta}(s_s') = s_t \cos 2\theta - 2s_s \sin 2\theta = 0 \text{ for maximum or minimum} \quad (1)$$

$$\tan 2\theta_s = \frac{s_t}{2s_s} = \frac{\frac{s_t}{2}}{s_s}. \quad (2)$$

The value of the maximum resultant unit shearing stress may be calculated by substituting in Equation (3) of the preceding article the values of cos 2θ and sin 2θ when tan $2\theta = \frac{\frac{s_t}{2}}{s_s}$.
To find cos 2θ and sin 2θ a right triangle may be formed with s_s as the base and $\frac{s_t}{2}$ as the altitude, Fig. 175. The angle adjacent to the side s_s is 2θ, the hypotenuse is

$$\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}.$$

$$\text{Cos } 2\theta = \frac{s_s}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}}, \quad \sin 2\theta = \frac{\frac{s_t}{2}}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}}. \quad (3)$$

When these values of cos 2θ and sin 2θ are substituted in the expression for s_s' and a common factor is divided out, the result is

$$\max s_s' = \pm \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}. \qquad \text{Formula XXXI.}$$

A comparison of the equations with Fig. 175 shows that the maximum resultant shearing stress is the hypotenuse of a right triangle of which the applied unit shearing stress is the base and one-half the applied unit tensile stress is the altitude. The angle between the maximum resultant shearing stress and the direction of the applied tension is one-half the angle which the hypotenuse of this triangle makes with the applied tension. The broken line through C in Fig. 175 gives the direction of one maximum shearing stress.

For any given tangent there are two angles which differ by 180 degrees; consequently there are two values of 2θ which are

180 degrees apart and two corresponding values of θ which are 90 degrees apart. These correspond to the two values of maximum shear at right angles to each other. The second maximum (or minimum) is along the broken line through D in Fig. 175.

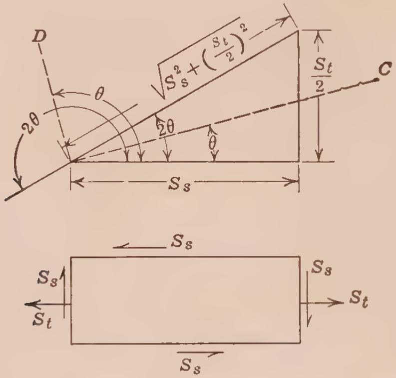


FIG. 175.—Maximum resultant shearing stress.

Example

A part of a solid is subjected to a horizontal tensile stress of 400 pounds per square inch and a horizontal and vertical shearing stress of 100 pounds per square inch. Find the direction and magnitude of the maximum resultant unit shearing stress.

$$\text{Tan } 2\theta = \frac{200}{100};\ 2\theta = 63°\ 26' \text{ or } 243°\ 26';$$

$$\theta = 31°\ 43' \text{ or } 121°\ 43'.$$

$$\text{Maximum } s_s' = 100\sqrt{5} = \pm 223.6.$$

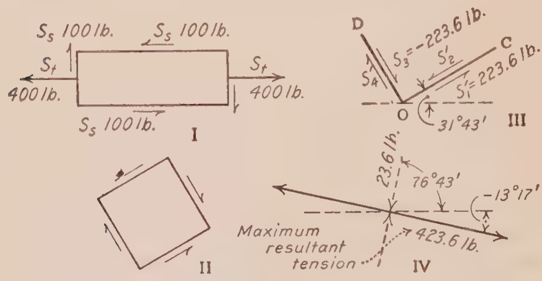


FIG. 176.

Figure 176, I, shows the applied tension and shear of this example. Fig. 176, II, shows the maximum resultant shearing stresses which act on a portion of the body from the material outside this portion. The magnitude of each of these four shearing stresses is 223.6 pounds per square inch.

Figure 176, II, shows the shearing stresses on only one side of each plane. In Fig. 176, III, the shearing stresses are shown on both sides of two of the planes which intersect at O. In Fig. 174, s_3' is the shearing stress which the material to the right of the plane $CBFG$ exerts on the material to the left of this plane, and the equations of Article 149 are based on this relation. In Fig. 176, III, s_1' is the shearing stress which the material to the right of the plane OC exerts on the material to the left of the plane. If the positive direction is from O toward C, s_1' is the *maximum* unit shearing stress. At the plane OD, at right angles to OC, s_3' is the stress which the material to the right of the plane exerts on the material to the left of the plane. Since this stress is opposite the positive direction of OD, it is taken as negative, and is, therefore, the *minimum*, which is represented by -223.6 pounds per square inch.

Problems

1. A part of a solid is subjected to a horizontal tensile stress of 600 pounds per square inch and a horizontal and a vertical shearing stress of 400 pounds per square inch. Find the direction and magnitude of the maximum unit shearing stress.
   Ans. $2\theta = 36° \ 52'$ or $216° \ 52'$; maximum unit shearing stress = 500 pounds per square inch at $18° \ 26'$ and at $108° \ 26'$ with the horizontal.
2. Find the maximum resultant shearing stress which is caused by a horizontal tensile stress of 400 pounds per square inch and a horizontal and vertical shearing stress of 300 pounds per square inch.
3. The maximum resultant of a horizontal tensile stress and of a horizontal and a vertical shearing stress is a shearing stress of 500 pounds per square inch at an angle of 25° with the horizontal. Find the applied tensile and shearing stresses.
4. The maximum resultant of a horizontal tensile stress of 240 pounds per square inch and a vertical and a horizontal shearing stress is a shearing stress of 350 pounds per square inch. Find the applied shearing stress and the direction of the maximum resultant shearing stress.
5. In the example above, find the resultant shearing stress at angles of 45°, 60°, 75°, 90°, and 120° with the horizontal. Also find the angle at which the resultant shearing stress is zero. *Ans.* $s_s' = 0$ at $55° \ 02'$.

151. Maximum Resultant Unit Tensile Stress.—From Equation (6) of Article 149

$$s_t' = \frac{s_t}{2}(1 - \cos 2\theta) + s_s \sin 2\theta. \tag{1}$$

$$\frac{d}{d\theta}(s_t') = s_t \sin 2\theta + 2s_s \cos 2\theta. \tag{2}$$

For the maximum and minimum s_t',

$$\tan 2\theta = -\frac{2s_s}{s_t} = -\frac{s_s}{\frac{s_t}{2}} \tag{3}$$

Comparison with Equation (2) of the preceding article shows that the double angle for maximum and mimimum tensile stress is normal to corresponding direction for maximum shear, and,

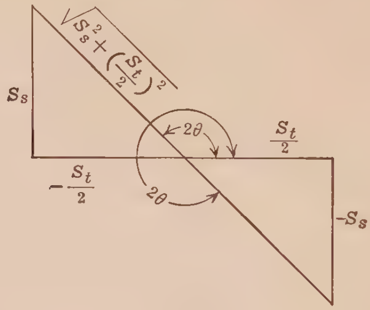


FIG. 177.—Double angle for maximum resultant tensile stress.

consequently, the directions of maximum and minimum tensions are at 45 degrees with the directions of maximum and minimum shear.

The double angle in the second quadrant, Fig. 177, gives

$$\sin 2\theta = \frac{s_s}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}}, \quad \cos 2\theta = -\frac{\frac{s_t}{2}}{\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}},$$

When these values of the sine and cosine of 2θ are substituted in Equation (1), the result is

$$\text{maximum } s_t' = \frac{s_t}{2} + \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2} =$$

$$\frac{s_t}{2} + \max s_s'. \quad \text{Formula XXXII.}$$

For the double angle in the fourth quadrant, the sine of 2θ is negative and the cosine is positive. When these are substituted in Equation (1), the result is

$$\text{minimum } s_t' = \frac{s_t}{2} - \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2} = \frac{s_t}{2} - \max s_s'. \tag{4}$$

Since the maximum unit shearing stress is always equal to or greater than one-half the unit tensile stress, the second term of (4) is never less than the first term and the minimum stress is compressive.

Example

Find the direction and magnitude of the maximum unit tensile stress for the example of the preceding article.

Tan $2\theta_t = -\frac{100}{200}$; $2\theta_t = -26°\ 34'$ or $153°\ 26'$; $\theta_t = -13°\ 17'$ or $76°\ 43'$.

Maximum $s_t' = 200 + 223.6 = 423.6$ pounds per square inch; minimum $s_t' = 200 - 223.6 = 23.6$ pounds per square inch compression. Fig. 176, III, shows this tension and compression.

Problems

1. Find the maximum resultant tensile stress for Problem 1 of the preceding article.
2. Find the maximum resultant shearing and tensile stress which are produced by a horizontal tensile stress of 1,000 pounds per square inch and a horizontal and vertical shearing stress of 600 pounds per square inch.
3. A body is subjected to a horizontal tensile stress of 600 pounds per square inch and to a horizontal and vertical shearing stress. The maximum resultant tensile stress is 800 pounds per square inch. What is the maximum resultant shearing stress? What is the maximum resultant compressive stress?
4. Find the direction of the maximum resultant tensile and shearing stresses in Problem 3.
5. Find the maximum resultant shearing and tensile stresses which are caused by a horizontal tension of 300 pounds per square inch and a horizontal and vertical shear of 160 pounds per square inch.

Ans. Tan $2\theta = 0.9375$; $\theta_s = 21°\ 35'$ or $111°\ 35'$.
Max $s_s' = 219.32$ lb./in.^{2}
Max $s_t' = 369.32$ lb./in.^{2}; min $s_t' = -69.32$ lb./in.^{2}

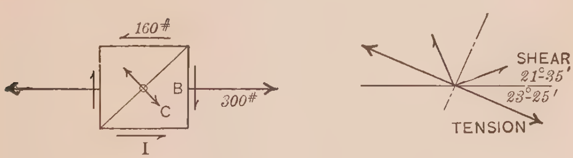


FIG. 178.—Direction of resultant tension.

Figure 178, which applies to Problem 5, indicates a method for finding whether the maximum tensile stress is 45° above or 45° below the direction of the maximum shearing stress. The tension which is caused by shear alone is 45° below the horizontal on the right side. The tension which results from this and the tensile stress of 300 pounds combined must lie between the two, and is, therefore, below the horizontal. The maximum shearing stress is 21° 35′ above the horizontal. The maximum tensile stress is 21° 35′ − 45° 00′ or 23° 25′ below the horizontal. The

minimum tensile stress, which is a compression of 69.32 pounds per square inch, lies in the direction of the broken line of Fig. 178.

152. The Resultant Stress in a Beam.—In a beam, the maximum resultant stress is due to a shearing stress which is a maximum at the neutral surface, and a tensile or compressive stress which is the greatest at the outer fibers. It is not usually necessary to calculate the maximum resultant tensile stress in a beam, since it is seldom greater than the bending stress in the outer fibers.

Problem

A 6-inch by 10-inch beam is supported at points 30 inches apart and carries a load of 20,000 pounds midway between the supports. Find the magnitude and direction of the maximum resultant tension, shear, and compression, at sections 5 inches and 10 inches from the left support at points 0, 1, 2, 3, 4, and 5 inches from the neutral axis.

Table XVI, below, gives the results of the calculation for this problem. It will be noticed that the tension is at 45 degrees with the horizontal at the neutral surface and is 250 pounds per square inch. At 5 inches from the end the resultant tensile stress increases to 500 pounds per square inch in the outer fibers, and at 10 inches from the end it increases to 1,000 pounds per square inch.

TABLE XVI.—RESULTANT SHEAR AND TENSION IN A BEAM

Distance below axis		Shear	Tension	Maximum shear		Maximum tension		Maximum compression	
		Pounds	Pounds	Pounds	Angle	Pounds	Angle	Pounds	Angle
At 5 inches from end	0	250	0	250.0	0° 0′	250.0	−45° 0′	250.0	45° 0′
	1	240	100	245.2	5° 53′	295.2	−39° 07′	195.2	50° 53′
	2	210	200	232.6	12° 44′	332.6	−32° 16′	132.6	57° 44′
	3	160	300	219.3	21° 35′	369.3	−23° 25′	69.3	66° 35′
	4	90	400	219.0	32° 53′	419.0	−12° 07′	19.0	77° 53′
	5	0	500	250.0	45° 0′	500.0	0° 0′	0	90° 0′
At 10 inches from end	0	250	0	250.0	0° 0′	250.0	−45° 0′	250.0	45° 0′
	1	240	200	260.2	11° 49′	360.2	−33° 11′	160.2	56° 49′
	2	210	400	290.0	21° 48′	490.0	−23° 12′	90.0	66° 48′
	3	160	600	341.0	30° 58′	641.0	−14° 2′	41.0	75° 58′
	4	90	800	410.0	38° 40′	810.0	− 6° 20′	10.0	83° 40′
	5	0	1,000	500.0	45° 0′	1,000.0	0° 0′	0	90° 0′

The shearing stress is 250 pounds per square inch at the neutral surface at both sections. At the outer fibers the shearing stress is entirely due to the tensile stress and is 250 pounds per square

inch at the section 5 inches from the support and 500 pounds per square inch at the section 10 inches from the support.

The tension at the neutral surface, which is entirely due to shear, is 250 pounds per square inch and makes an angle of 45° with the length of the beam. The maximum tension in the outer fibers is the same as if there were no shear.

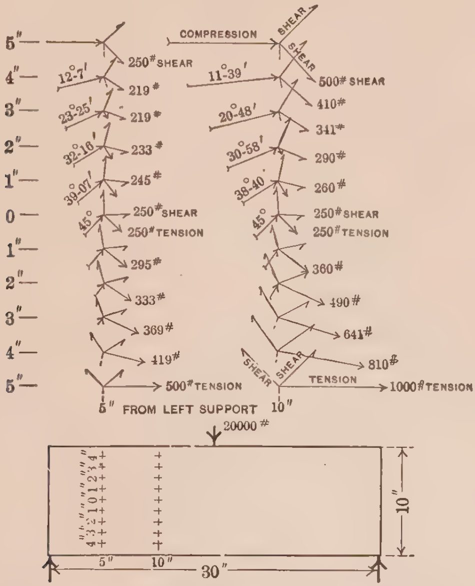


FIG. 179.—Resultant stress in a beam section.

Since the shear is a maximum at the neutral surface, where the bending stress is zero, and the tension is a maximum in the outer fibers, where the shear is zero, the maximum resultant stress at any point in a beam is seldom greater than the stress in the outer fibers which is due to bending alone. In a short I-beam, it may happen that the resultant tension in the web at the connection with the flange may slightly exceed the stress in the outer fibers. An 18-inch, 42-pound I-beam, for instance may have a unit bending stress of 13,500 pounds per square inch in the outer fibers. At one inch from the top, the bending stress is $\frac{8}{9} \times 13{,}500 = 12{,}000$ pounds per square inch. If the total vertical shear is 41,400 pounds, the average unit shearing stress

is $\frac{41{,}400}{18 \times 0.46} = 5{,}000$ pounds per square-inch. The shearing stress 8 inches from the neutral surface is a little less than 5,000 pounds per square inch, but may be assumed to have that value.

$$\text{Max } s_t' = 6{,}000 + 1{,}000\sqrt{36 + 25} = 13{,}810 \text{ lb./in.}^2$$

which is very little more than the bending stress in the outer fibers.

In a reinforced concrete beam, the steel in tension is mathematically equivalent to a very wide flange of concrete. The unit shearing stress in the concrete which adjoins the reinforcement is large. Since the steel is near the outer surface, the bending stress in the concrete is also large. The resultant tensile stress in the concrete is relatively large, and such beams often begin to crack along surfaces at right angles to the direction of the maximum resultant tension.

153. Bending Combined with Torsion.—In a shaft subjected to bending moment, the maximum tensile stress is found in the fibers at the dangerous section which are most remote from the neutral surface. When subjected to torsion, all the outer fibers are at the maximum shearing stress. When the shaft is subjected to the combined effect of bending moment and torque, those fibers at the dangerous section which are farthest from the neutral surface are subjected to the combined effect of the maximum tensile or compressive stress and the maximum shearing stress which may be much larger than the results of Formulas VII and XIV.

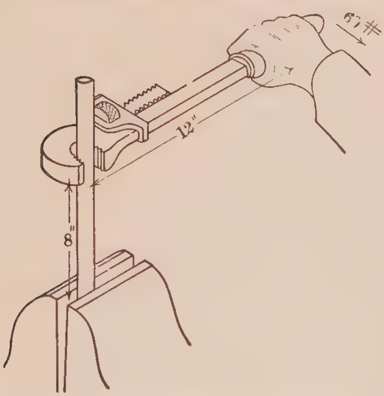


FIG. 180.—Torsion and bending.

Example

A 1-inch rod projects from a vise. A wrench, at right angles to the rod, grips it 8 inches from the vise. The wrench is turned by a force of 60 pounds, perpendicular to the plane of the rod and wrench, which is applied to the wrench 12 inches from the axis of the rod. Find the maximum resultant shearing and tensile stress.

The bending moment at the vise is the same as if the force of 60 pounds were applied directly to the rod at 8 inches from the vise, Fig. 180.

$$M = 60 \times 8 = 480 \text{ inch-pounds.}$$
$$S_t = \frac{480 \times 32}{\pi} = \frac{3{,}840 \times 4}{\pi} = 4{,}889 \text{ lb./in.}^2$$
$$T = 60 \times 12 = 720 \text{ inch-pounds.}$$
$$S_s = \frac{720 \times 16}{\pi} = \frac{3{,}840 \times 3}{\pi} = 3{,}667 \text{ lb./in.}^2$$
$$\text{Maximum } S_s' = \sqrt{3{,}667^2 + 2{,}444^2} = 4{,}407 \text{ lb./in.}^2$$
$$\text{Maximum } S_t' = 2{,}444 + 4{,}407 = 6{,}851 \text{ lb./in.}^2$$

Since the section modulus used in torsion is twice that used in bending, and the force P is the same for both torque and bending moment, there is a large common factor which may be taken out to reduce the labor of computation. In this problem the factor is $\frac{3{,}840}{\pi}$ which is equal to 1,222.

$$\text{Max } S_s' = 1{,}222\sqrt{3^2 + 2^2} = 1{,}222\sqrt{13} = 4{,}407 \text{ lb./in.}^2$$

Problems

1. A 2-inch rod, which projects from a vise, is twisted by a pipe wrench, which is applied 20 inches from vise. A force of 360 pounds at right angles to the plane of the rod and the vise is applied 40 inches from the axis of the rod. Find the maximum unit shearing and tensile stress in the rod.

Ans. Max $S_s' = 10{,}249$ lb./in.^{2}; max $S_t' = 14{,}833$ lb./in.^{2}

2. A 4-inch solid shaft transmits 200 hp. at 150 r.p.m., and is subjected to a compression of 40,000 pounds in the direction of its length. Find the maximum resultant compressive and shearing stress.

Ans. Max $S_s' = 6{,}874$ lb./in.^{2}; max $S_c' = 8{,}465$ lb./in.^{2}

154. Equivalent Moment and Torque.—For a circular section $J = 2I$, and, when the outer radius is a,

$$S_t = \frac{Ma}{I}, \quad \frac{S_t}{2} = \frac{Ma}{2I}; \tag{1}$$

$$S_s = \frac{Ta}{J} = \frac{Ta}{2I}, \tag{2}$$

$$\text{Max } S_s' = \sqrt{S_s^2 + \left(\frac{S_t}{2}\right)^2} = \frac{a}{2I}\sqrt{T^2 + M^2} = \frac{a\sqrt{T^2 + M^2}}{J}. \tag{3}$$

The term $\sqrt{T^2 + M^2}$ may be regarded as the equivalent torque resulting from the combination of torsion and bending. In the example of the preceding article $M = 480$ inch-pounds,

$T = 720$ inch-pounds and the equivalent torque is $240\sqrt{13} = 865.3$

$$\text{Max } S_s' = \frac{865.3 \times 16}{\pi} = 4{,}407 \text{ lb./in.}^2$$

$$\text{Max } S_t' = \frac{Ma}{2I} + \frac{a\sqrt{T^2 + M^2}}{2I} = a\frac{(M + \sqrt{T^2 + M^2})}{2I}. \qquad (4)$$

The term $\dfrac{M + \sqrt{T^2 + M^2}}{2}$ may be regarded as the equivalent bending moment.

Problem

A hollow shaft of 4 inches inside diameter and 6 inches outside diameter is subjected to a torque of 2,000 foot-pounds and a bending moment of 1,500 foot-pounds. Find the equivalent maximum torque and moment and find the maximum unit shearing and tensile stress.

155. Shear Combined with Tension in Two Directions.—Figure. 181 represents a block subjected to shearing stress and a horizontal tension as in Fig. 174 together with a vertical tension of intensity s_v. To find the intensity of the resultant unit shearing and tensile stresses in any direction, the block is divided by a plane which makes an angle θ with the horizontal, and the wedge to the left of this plane is taken as the free body. The shearing stress at an angle θ with the horizontal is found by resolving parallel to this plane. If the length of the block perpendicular to the plane of the paper is taken as unity, the resolution is

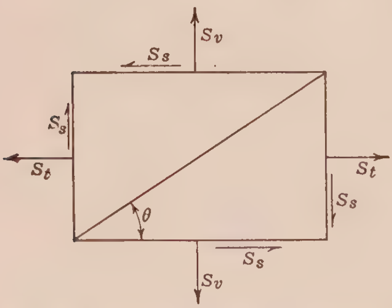


FIG. 181.—Tension in two directions combined with shear.

$$s_s'ds = s_t dy \cos\theta + s_s dx \cos\theta - s_s dy \sin\theta - s_v dx \sin\theta. \qquad (1)$$

When Equation (1) is divided by ds, and $\cos\theta$ is substituted for $\dfrac{dx}{ds}$ and $\sin\theta$ is substituted for $\dfrac{dy}{ds}$, the result is

$$s_s' = (s_t - s_v)\sin\theta\cos\theta + s_s(\cos^2\theta - \sin^2\theta); \qquad (2)$$

$$s_s' = \frac{s_t - s_v}{2}\sin 2\theta + s_s\cos 2\theta. \qquad (3)$$

Equation (3) is the same as Equation (3) of Article 149 except that $s_t - s_v$ replaces s_t.

To get the direction of the maximum unit shearing stress, Equation (3) is differentiated with respect to θ, and the derivative equated to zero.

$$(s_t - s_v) \cos 2\theta - 2s_s \sin 2\theta = 0; \tag{4}$$

$$\tan 2\theta = \frac{\frac{s_t - s_v}{2}}{s_s}. \tag{5}$$

When the values of the sine and cosine of θ from Equation (5) are substituted in Equation (3), the result is

$$\text{Max } s_s' = \sqrt{s_s^2 + \left(\frac{s_t - s_v}{2}\right)^2}, \tag{6}$$

which is the same as Formula XXXI with $s_t - s_v$ in place of s_t.

To find the resultant unit tensile stress across the inclined plane, a resolution is taken perpendicular to it. The equation of equilibrium is

$$s_t' \, ds = s_t dy \sin\theta + s_s dx \sin\theta + s_s dy \cos\theta + s_v dx \cos\theta; \tag{7}$$

$$s_t' = s_t \sin^2\theta + s_v \cos^2\theta + 2\, s_s \sin\theta \cos\theta; \tag{8}$$

$$s_t' = \frac{s_t}{2}(1 - \cos 2\theta) + \frac{s_v}{2}(1 + \cos 2\theta) + s_s \sin 2\theta. \tag{9}$$

For the direction of maximum unit tensile stress,

$$\tan 2\theta = -\frac{s_s}{\frac{s_t - s_v}{2}}, \tag{10}$$

which is normal to the double angle for the maximum shearing stress.

$$\text{Max } s_t' = \frac{s_t + s_v}{2} + \sqrt{s_s^2 + \left(\frac{s_t - s_v}{2}\right)^2}. \tag{11}$$

If the vertical stress is compressive it may be regarded as negative tension. If s_c is this compressive stress,

$$\max s_s' = \sqrt{s + \left(\frac{s_t + s_c}{2}\right)^2}; \tag{12}$$

$$\max s_t' = \frac{s_t - s_c}{2} + \max s_s'. \tag{13}$$

If s_s is zero, Equation (6) gives $\frac{s_t - s_v}{2}$ as the maximum unit shearing stress at 45 degrees with both s_t and s_v. There is, however, a greater unit shearing stress of magnitude $\frac{s_t}{2}$ in a plane parallel to s_v, at an angle of 45 degrees with the direction of the greater stress, s_t.

Equation (11) shows that when the shearing stress is zero, the maximum tensile stress is s_t.

Problems

1. A block is subjected to a horizontal tensile stress of 600 pounds per square inch, vertical compressive stress of 200 pounds per square inch, and horizontal and vertical shearing stress of 300 pounds per square inch. Find the maximum unit shearing and tensile stress.

Ans. Max $s_s' = 500$ lb./in.^{2}; max $s_t' = 700$ lb./in.^{2}

2. A 1-inch round rod projects from a vise and is twisted by a force of 60 pounds at the end of a 12-inch wrench. The pressure at the jaws of the vise is 4,000 pounds per square inch. Find the maximum stresses if the wrench is applied 10 inches from the vise and the direction of the force of 60 pounds is normal to the plane of the jaws.

Ans. Max $s_s' = 6{,}245$ lb./in.^{2}; max $S_t' = 7{,}300$ lb./in.^{2}

3. A block is subjected to a horizontal tensile stress of 600 pounds per square inch, and a vertical tensile stress of 200 pounds per square inch, together with horizontal and vertical shearing stress of 300 pounds per square inch in the plane of the two tensile stresses. Find the maximum unit shearing stress. *Ans.* Max $s_s' = 360.6$ pounds per square inch.

4. Solve Problem 3 if the unit shearing stress be only 100 pounds per square inch. *Ans.* Max $s_s' = 223.6$ pounds per square inch.

CHAPTER XVI

RESILIENCE IN BENDING AND SHEAR

156. Resilience in Bending.—It was shown in Article 12 that the elastic energy per cubic inch is $\frac{s^2}{2E}$ and that the total energy in a body is the energy per cubic inch multiplied by the volume of the body. The unit stress in a beam varies as the distance from the neutral axis and also varies as the moment at the section. The total elastic energy may be determined in either of two ways: (1) The total work of the *external* forces may be calculated. (2) An expression may be written for the *internal* unit stress s, and this expression may be integrated over the entire volume of the beam.

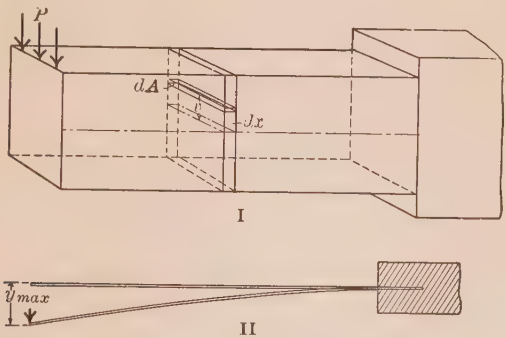


FIG. 182.—Work of deflection.

157. Resilience by External Work.—If a load P causes a deflection $y_{\max}$ at its point of application, the work is $\frac{Py_{\max}}{2}$. In a cantilever with a single concentrated load, Fig. 182,

$$y_{\max} = \frac{Pl^3}{3EI},$$

$$\text{external work} = \frac{P^2l^3}{6\,EI}. \tag{1}$$

In a beam supported at the ends with a load at the middle,

$$\text{external work} = \frac{P}{2} \times \frac{Pl^3}{48EI} = \frac{P^2l^3}{96EI}. \tag{2}$$

When there is a uniformly distributed load of w pounds per unit length, the increment of load on a length dx is wdx, and the work done by this increment is $\frac{wydx}{2}$, in which y is the deflection of the particular part of the beam upon which the increment rests. In Fig. 183, II, one increment wdx is deflected a distance

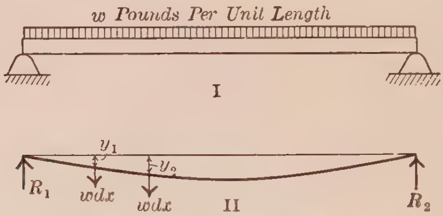


FIG. 183.—External work.

y_1, a second increment is deflected a distance y_2, and so forth.

The different values of y are determined from the equation of the elastic line. The total work is the sum of these increments of work.

$$\text{Total work} = \frac{w}{2}\int ydx, \tag{3}$$

with the ends of the beam as the limits.

For a cantilever with a uniformly distributed load,

$$y = \frac{w}{24EI}(3l^4 - 4l^3x + x^4) \tag{4}$$

if y is taken as positive downward. (If y is taken positive upward in the usual way, the deflection is negative. The load w of $w\,dx$ is also negative and the work is positive.) The work of an increment $w\,dx$ is

$$dU = \frac{w^2}{48EI}(3l^4 - 4l^3x + x^4)dx; \tag{5}$$

$$U = \frac{w^2}{48EI}\left[3l^4x - 2l^3x^2 + \frac{x^5}{5}\right]_0^l = \frac{w^2l^5}{40EI}. \tag{6}$$

Problems

1. Find the total external work on a uniform beam which is supported at the ends and carries a uniformly distributed load.

Ans. $U = \frac{w^2l^5}{240EI}$.

2. A beam of length l is supported at the ends and carries a load P at a distance a from one end. If $l - a = b$, find the external work in terms of a, b and l.

Ans. $U = \dfrac{P^2a^2b^2}{6EIl}$.

158. Expression for Internal Work.—The unit stress at a distance v from the neutral axis of a beam is $\dfrac{Mv}{I}$. Figure 182, I, shows an element of volume of cross-section dA and length dx at a distance v from the neutral axis. The energy in this element of volume $dA\ dx$ is

$$dU = \frac{s^2}{2E}\,dAdx = \frac{M^2v^2}{2EI^2}\,dAdx. \tag{1}$$

$$\text{Total work in beam} = \int\!\!\int \frac{M^2}{2EI^2} v^2 dAdx. \tag{2}$$

The integration of Equation (2) with respect to v as the variable gives the work done upon the volume of length dx between two vertical planes. Throughout this volume, x, M, and I are constant. The integral of v^2dA across the beam from the bottom to the top is I.

$$\text{Work} = \int \frac{M^2}{2EI} dx. \tag{3}$$

Equation (3) may be used to calculate the internal work in any beam. Unless M and I are constant, they must be expressed as functions of x before integrating.

159. Internal Work in a Beam under Constant Moment.—In a beam of uniform section, when the bending moment is constant,

$$U = \frac{M^2}{2EI}\int dx = \frac{M^2}{2EI}\Big[x\Big]_0^l = \frac{M^2l}{2EI}. \tag{1}$$

Equation (1) gives the total work in terms of the bending moment. It is often desirable to find the work in terms of the unit stress in the extreme fibers. If the neutral axis is midway between the extreme top and bottom fibers, $c = \dfrac{d}{2}$, and $M = \dfrac{2SI}{d}$. When this value of the moment is substituted in Equation (1), the result is

$$U = \frac{4S^2I^2l}{2EId^2} = \frac{2S^2Il}{Ed^2}. \tag{2}$$

For a rectangular section,

$$U = \frac{2S^2bd^3l}{12Ed^2} = \frac{S^2bdl}{6E} = \frac{S^2}{6E} \times \text{volume}. \tag{3}$$

The average energy per unit of volume is $\frac{S^2}{6E}$, which is one-third as great as that in a block subjected to a uniform stress S.

Problems

1. Find the average energy per unit volume in a solid circular section subjected to a uniform bending moment.

Ans. $\frac{S^2}{8E}$.

2. A steel bar 2 inches wide and ½ inch thick is 8 feet long and rests on two supports 6 feet apart and carries two equal loads on the ends. If E is 30,000,000 pounds per square inch, what is the total elastic energy in the part between the supports when each load on the ends is 100 pounds?

Ans. 82.9 inch-pounds.

160. Internal Work in a Beam under a Uniformly Distributed Load.—For a cantilever with uniformly distributed load with the origin of coördinates at the free end, the moment is $\frac{wx^2}{2}$ and

$$U = \frac{w^2}{8E}\int_0^l \frac{x^4}{I}dx. \tag{1}$$

When I is constant

$$U = \frac{w^2l^5}{40EI} = \frac{W^2l^3}{40EI}. \tag{2}$$

For a section which is symmetrical with respect to the neutral axis

$$\frac{Wl}{2} = \frac{2SI}{d},$$

$$U = \frac{4S^2Il}{10Ed^2}. \tag{3}$$

For a rectangular section, for which $I = \frac{bd^3}{12}$,

$$\text{total work} = \frac{S^2bdl}{30E} = \frac{S^2}{30E} \times \text{volume}. \tag{4}$$

The total energy in a cantilever of rectangular section with uniformly distributed load is one-fifteenth as much as that in a block of the same volume with uniform compressive stress throughout.

For a *beam supported at the ends with uniformly distributed load,* $M = \frac{wlx}{2} - \frac{wx^2}{2}$.

$$U = \frac{w^2}{8EI}\int (l^2x^2 - 2lx^3 + x^4)dx, \tag{5}$$

$$U = \frac{w^2}{8EI}\left[\frac{l^2x^3}{3} - \frac{lx^4}{2} + \frac{x^5}{5}\right]_0^l = \frac{w^2l^5}{240EI} \tag{6}$$

which agrees with the answer to Problem 1, of Article 157.

Problem

Find the work per unit volume in terms of the maximum unit stress in a beam of rectangular section which is supported at the ends and uniformly loaded.

Ans. $\frac{4S^2}{45E}$.

161. Internal Work in a Beam with a Single Concentrated Load.—For a cantilever with a load on the free end, $M = -Px$. When the section is constant,

$$U = \frac{P^2}{2EI}\int x^2dx = \frac{P^2}{6EI}\Big[x^3\Big]_0^l = \frac{P^2l^3}{6EI}. \tag{1}$$

For a beam supported at the ends with a load P at a distance a from one end and at a distance b from the other, the reaction at the end of the length a is $\frac{Pb}{l}$ and the moment in this length is $\frac{Pbx}{l}$. The work in this part of the beam is,

$$U = \frac{P^2b^2}{2EIl^2}\int_0^a x^2dx = \frac{P^2b^2a^3}{6EIl^2}. \tag{2}$$

Similarly in the length b,

$$\text{Work} = \frac{P^2a^2b^3}{6EIl^2}. \tag{3}$$

For the entire length,

$$\text{Total work} = \frac{P^2a^2b^2(a+b)}{6EIl^2} = \frac{P^2a^2b^2}{6EIl}. \tag{4}$$

When the load is at the middle, $a = b = \frac{l}{2}$ and

$$\text{total work} = \frac{P^2l^3}{96EI}. \tag{5}$$

Problems

1. Check Equation (5) by external work.
2. Find the elastic energy per unit volume in terms of the maximum unit stress, for a cantilever of rectangular section with a load on the free end.

   Ans. $\frac{S^2}{18E}$.
3. Solve Problem 2 for a beam which is supported at the ends and loaded at the middle.
4. Solve Problem 2 for a beam which is supported at the ends and loaded at any point.
5. Find the elastic energy per unit volume in a hollow cylindrical cantilever in terms of the maximum unit stress.

   Ans. $U = \frac{S^2(R_2^2 + R_1^2)}{24R_2^2E}$, in which R_2 is the outside radius and R_1 is the inside radius.

162. Beam of Variable Section.—In a beam of variable section, the moment of inertia is a variable. At a distance v from the neutral axis of any beam, $s = kv$, in which k is a constant depending upon the external moment and upon the form of the section.

$$\text{Work per unit volume} = \frac{k^2v^2}{2E}. \tag{1}$$

The element of volume for a rectangular section of breadth b is $b\,dv\,dx$. For the integration with respect to v, the terms k and b are constant.

$$U = \frac{1}{2E}\int bk^2 \int_{-\frac{d}{2}}^{\frac{d}{2}} (v^2dv)dx = \frac{1}{24E}\int bk^2d^3dx \tag{2}$$

Since $\frac{kd}{2}$ is equal to S, the maximum fiber stress,

$$U = \frac{1}{6E}\int S^2bd\,dx. \tag{3}$$

If the beam is so designed that S is constant for all sections,

$$U = \frac{S^2}{6E}\int bd\,dx = \frac{S^2}{6E} \times \text{volume}. \tag{4}$$

The elastic energy per unit volume in a beam of constant strength and rectangular section is $\frac{S^2}{6E}$, which is one-third as great as that in a block subjected to uniform compressive stress equal to S.

163. Calculation of Deflection by Internal Work.—The work done in a beam affords a method of finding the deflection under a concentrated load. If P is the load and y is the deflection, the external work is $\frac{Py}{2}$. When this expression is equated to the internal work, the value of the deflection may be found in terms of the load. In Article 161, for instance, the internal work caused by a single concentrated load on a beam supported at the ends was found to be $\frac{P^2a^2b^2}{6EIl}$. To find the deflection under the load

$$\frac{Py}{2} = \frac{P^2a^2b^2}{6EIl}; \qquad (1)$$

$$y = \frac{Pa^2b^2}{3EIl}. \qquad (2)$$

Problems

1. A 4-inch by 6-inch wooden beam, 20 feet long, is supported 5 feet from each end and carries a load of 200 pounds on each end. Calculate the total internal work, and find the deflection at each end, if $E = 1{,}000{,}000$ pounds per square inch. *Ans.* Deflection at each end = 0.8 inch.
2. A 6-inch by 2-inch plank, 20 feet long, is supported 5 feet from the left end and held down at the left end. A load of 20 pounds is placed at the right end. Find the deflection if $E = 1{,}200{,}000$ pounds per square inch. *Ans.* 10.8 inches.

164. Deflection of Beam of Constant Strength by Internal Work.—For a cantilever of constant strength, constant depth, and rectangular section, with load on the free end, the breadth varies as the distance from the free end. The volume is $\frac{BDl}{2}$, in which B is the maximum breadth and D is the depth.

$$\text{Total work} = \frac{S^2}{6E} \times \frac{BDl}{2} = \frac{Py_{\max}}{2}. \qquad (1)$$

The maximum fiber stress, from the dimensions at the fixed end, is

$$S = \frac{PlD}{2I_m}, \qquad (2)$$

in which I_m is the moment of inertia at the fixed end.

$$\text{Total work} = \frac{P^2BD^3l^3}{48EI_m^2} = \frac{Py_{\max}}{2}. \qquad (2)$$

Since

$$I_m = \frac{BD^3}{12},$$

$$\frac{P^2l^3}{4EI_m} = \frac{Py_{\max}}{2}, \tag{3}$$

$$y_{\max} = \frac{Pl^3}{2EI_m}.$$

(Compare with Article 119.)

In a rectangular cantilever beam of constant strength and constant breadth with a load on the end, the depth is given by the equation of the parabola

$$d^2 = \frac{6Px}{SB}.$$

The area is $\frac{2Dl}{3}$ and the volume is $\frac{2BDl}{3}$. Making the same substitutions as in the preceding case,

$$y_{\max} = \frac{2Pl^3}{3EI_m}. \tag{4}$$

Problem

Find the deflection at the end of a cantilever of constant strength and square section due to a load on the end.

Ans. $y_{\max} = \frac{3Pl^3}{5EI_m}$.

165. Internal Work in a Shaft.—The unit shearing stress s_s produces a deformation of $\frac{s_s}{E_s}$ in planes at unit distance apart. The work of shear is the product of half the unit stress by the total deformation,

$$\text{work per unit volume} = \frac{s_s}{2} \times \frac{s_s}{E_s} = \frac{s_s^2}{2E_s}, \tag{1}$$

In a solid circular shaft at a distance r from the axis, the unit shearing stress is kr and

$$\text{energy per unit volume} = \frac{k^2r^2}{2E_s}, \tag{2}$$

The element of volume of length l is $2\pi rldr$ and

$$\text{total energy} = \frac{\pi k^2 l}{E_s}\int^a r^3dr = \frac{\pi lk^2a^4}{4E_s}, \tag{3}$$

in which a is the radius of the shaft. The maximum unit shearing stress in the outer surface is $S_s = ka$ and

$$\text{total energy of shear} = \frac{S^2}{4E_s}\pi a^2 l = \frac{S^2}{4E_s} \times \text{volume}. \tag{4}$$

Since the modulus of elasticity in shear is about two-fifths as great as the modulus in tension or compression, the total energy of a rod in torsion, for equal values of the unit stress, is one-fourth greater than that of the same rod in tension. However, since the elastic limit of steel and other similar materials in shear is somewhat smaller than in tension, the total energy which may be stored is approximately the same in both cases.

166. Work of Shear in a Rectangular Beam.—In a beam of rectangular section of breadth b and depth d subjected to a vertical shear V

$$s_s = \frac{V}{Ib}\int bv\,dv = \frac{V}{I}\left[\frac{v^2}{2}\right]_v^{\frac{d}{2}} = \frac{V}{8I}(d^2 - 4v^2); \tag{1}$$

$$\frac{s_s^2}{2E_s} = \frac{V^2}{128E_sI^2}(d^4 - 8d^2v^2 + 16v^4). \tag{2}$$

When the second member of Equation (2) is multiplied by the element of volume, which is $b\,dv\,dx$, and the product is integrated with respect to v with the limits of $-\frac{d}{2}$ and $\frac{d}{2}$, the result is

$$U = \int \frac{V^2bd^5}{128E_sI^2}\left(1 - \frac{2}{3} + \frac{1}{5}\right)dx = \int \frac{3V^2}{5E_sbd}\,dx. \tag{3}$$

When V is constant, the last term of Equation (3) for a beam of constant section is $U = \frac{3V^2l}{5E_sbd}$. For a cantilever beam with a load on the free end, $V = -P$ and

$$U = \frac{3P^2l}{5E_sbd}. \tag{4}$$

To find the deflection which is caused by shear at the end of a cantilever with a load on the end,

$$\frac{Py}{2} = \frac{3P^2l}{5E_sbd},$$

$$y = \frac{1.2Pl}{E_sbd} = \frac{1.2\,Vl}{E_sbd} = \frac{1.2s_s'l}{E_s} \tag{5}$$

in which s_s' is the average unit shearing stress.

The same relation holds for a beam supported at the ends with a concentrated load at the middle.

167. Sections of Maximum Resilience.—To obtain the maximum resilience per unit volume, the stress in all parts of the solid should be the maximum allowable unit stress. This condition of maximum efficiency can be secured only when the material is used in direct tension or compression. On account of the small displacements and the large forces, it is not practicable to use any solid material, except soft rubber, as a spring in direct tension or compression.

Since only the outer fibers reach the maximum unit stress when the material is subjected to bending or torsion, the energy per unit volume must always be less than $\frac{S^2}{2E}$. In a rectangular beam of constant strength or in a rectangular beam of uniform section subjected to constant moment, the energy per unit volume is one-third as great as it would be if it were subjected to the maximum stress throughout, and is three times as great as the energy of a beam of uniform section which is supported at the ends and loaded at the middle.

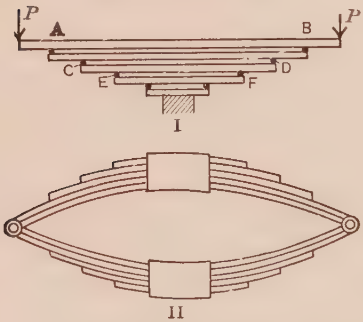


FIG. 184.—Leaf spring.

The common leaf springs (Fig. 184, II) are beams of constant strength made up of separate parts or leaves, each of which is subjected to constant bending moment throughout most of its length. In Fig. 184, I, the leaves are shown straight with each leaf resting on a pair of supports on the ends of the leaf below. In the upper leaf, the moment is constant from A to B, and the energy in that portion is $\frac{S^2}{6E}$. The overhanging parts act as cantilevers loaded at the ends, and the energy per unit volume is only $\frac{S^2}{18E}$. Between C and D, the moment is constant in the second leaf. If the distance from the load P to A is equal to the distance from A to C the moment at C is equal to the moment at A and, consequently, the moment between C and D in the second leaf must be equal to the moment between A and

B in the first leaf. If the overhang of each leaf is the same, and if the contacts are at the ends, as in Fig. 184, I, the unit stress in each leaf between the points of contact with the leaf below, is constant. In the actual leaf spring, as shown in Fig. 184, II, the contact takes place over a considerable area and the stresses and deflections are modified by friction.

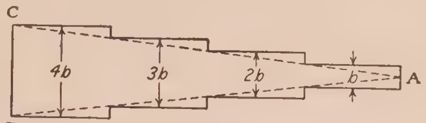


FIG. 185.

A leaf spring is approximately equivalent to a spring of constant thickness and variable width. Figure 185 shows a cantilever which may represent one-half of a spring of four leaves. The broken lines show a beam of constant strength for which the stress is equal to the maximum stress in the actual spring. The deflection of the beam which is represented by the broken lines is somewhat less than that of the spring. The deflection of a cantilever of constant depth and triangular plan which is loaded at the end is

$$y_{\max} = -\frac{Pl^3}{2EI_m},$$

in which I_m is the maximum moment of inertia. For a leaf spring, I_m is the moment of inertia of a rectangle of width equal to the total width of the leaves and thickness equal to that of a single leaf.

Since a relatively large part of an I-beam section is in the flange, where the unit stress approximates the maximum, the energy per unit volume is greater than in a rectangular section.

In a solid circular section in torsion, the energy per unit volume was shown in Article 47 to be $\frac{S^2}{4E_s}$. In a hollow shaft of inside radius b and outside radius a,

$$\text{total energy} = \frac{\pi k^2 l}{E_s}\int r^3 dr = \frac{\pi k^2 l}{4E_s}\Big[r^4\Big]_b^a \tag{1}$$

$$\text{total energy} = \frac{\pi k^2 l(a^4 - b^4)}{4E_s} \tag{2}$$

Since the volume is $\pi(a^2 - b^2)l$ and $ka = S_s$,

$$U = \frac{k^2(a^2 + b^2)}{4E_s} \times \text{volume} = \frac{(a^2 + b^2)S_s^2}{4a^2E_s} \times \text{volume} \tag{3}$$

Problems

1. What is the energy per unit volume of a hollow cylinder in torsion if the inside diameter is three-fourths of the outside diameter? *Ans.* $\frac{25S_s^2}{64E_s}$.
2. What is the total elastic energy of torsion in a hollow steel rod 5 feet long, 1 inch outside diameter and ½ inch inside diameter, when the unit shearing stress is 80,000 pounds per square inch and E_s is 12,000,000 pounds per square inch? *Ans.* 5,890.5 inch-pounds.
3. A hollow rectangular beam is 6 inches by 8 inches outside and 4 inches by 4 inches inside. Find the energy per unit volume if the external moment is constant throughout the length. *Ans.* $\frac{11S^2}{48E}$.
4. A spring, 48 inches long, is made of 6 leaves, each 2.5 inches wide and 0.1 inch thick. The load on each end of the spring is 300 pounds. Find the maximum fiber stress, and find the deflection. What effect has lubrication upon the strength of the longest leaf of a spring subjected to impact? What effect does it have on the other leaves?

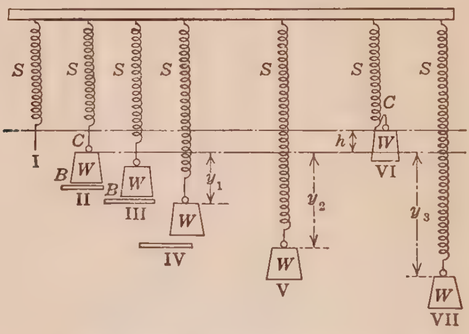


FIG. 186.—Effect of sudden loads and impact.

168. Impact Stresses.—In the discussion of resilience, it has been assumed that the load is applied so gradually that the average force is one-half the sum of the initial and final loads. If the initial load is zero and the final load P, the average load is $\frac{P}{2}$ and if the displacement under the load is y, the work is $\frac{Py}{2}$.

There are several ways of applying a load to meet these conditions. A load may be made up of a number of small parts and applied little at a time. For instance, a vessel may be hung from a beam or spring and gradually filled with water or sand. In testing large floors, bags of sand are added one at a time. In testing machines, the loads are applied gradually by means of slowly moving screws.

Figure 186 represents different ways of applying a single load.

Figure 186, I, shows a single spring. In Fig. 186, II, the same spring is shown with a mass W attached but with the entire weight of W carried by the support B. In Fig. 186, III, the support has been lowered. Part of the weight W is now carried by the spring and the remainder by the support B. In Fig. 186, IV, the support has been entirely removed and the mass W is at rest with the spring stretched a distance y_1. When the elongation of the spring was one-half of y_1, the spring carried one-half the weight and the support carried the other half. As the elongation increased, the portion of the load carried by the spring gradually increased and the portion carried by the support gradually decreased. The average load on the spring was $\frac{W}{2}$ and the average load on the support was the same. The total work of gravity in moving the weight W the distance y_1 was wy_1. Half of this work was expended in stretching the spring and the other half in assisting the motion of the support B.

If K is the force required to stretch the spring unit distance, the force required to stretch it a distance y_1 is Ky_1, which is equal to the weight of W; the energy stored in the spring is $\frac{Ky_1^2}{2}$.

If the support B is suddenly removed from W in the position II, the entire force of gravity is effective throughout the whole distance. At first the spring offers no resistance and the entire load goes to accelerate the mass (provided the mass of the spring is negligible). As it is stretched, the resistance of the spring increases. At the position IV the pull of the spring is equal to the weight and the acceleration is zero. The mass has its highest velocity at the point where it would come to rest under a gradually applied load. Beyond this point, represented by IV, the upward pull of the spring is greater than the weight and the body is negatively accelerated. It finally stops at the position of Fig. 186, V, at which the elongation of the spring is y_2. Since all the work of gravity has been expended in stretching the spring, the magnitude of the elongation y_2 may be calculated by equating this work to the energy of the spring.

$$Wy_2 = \frac{Ky_2^2}{2}, \tag{1}$$

$$y_2 = \frac{2W}{K} = 2y, \tag{2}$$

$$Ky_2 = 2W. \tag{3}$$

When a load is suddenly applied, the maximum force is twice the weight of the load, and the deflection is twice as great as it would be if the same load were applied gradually. After reaching the maximum elongation, the body vibrates back to the original starting point (provided the spring is perfectly elastic).

Figure 186, VI, shows the mass W lifted a distance h above the position of II, in which it exerts no pull on the spring. If released suddenly, it falls this distance before it begins to stretch the spring. The total work done by gravity is the weight multiplied by the total distance $h + y$. At the lowest position VII this work has been transformed to energy of the spring.

$$W(h + y) = \frac{Ky^2}{2}. \tag{4}$$

Problems

1. A force of 8 pounds stretches a given spring 1 foot. A 6-pound mass is placed on the spring and gradually lowered. How much will the spring be stretched? *Ans.* 9 inches.
2. In Problem 1 the load is applied suddenly. What is the elongation of the spring and the maximum pull? *Ans.* 18 inches; 12 pounds.
3. In Problem 1 the load is lifted 1.44 feet and suddenly released. Find the maximum elongation and the maximum pull. *Ans.* 2.4 feet; 19.2 pounds.
4. A spring board is made of plank 12 inches wide and 2 inches thick, and acts as a cantilever 10 feet long. Find the maximum unit stress when a boy weighing 60 pounds moves very slowly from the support to the free end. *Ans.* 900 pounds per square inch.
5. Solve Problem 4 if the boy steps suddenly on the free end, neglecting the effect of the mass of the spring board upon its vibrations. *Ans.* 1,800 pounds.
6. Solve problem 5 if the boy jumps down on the end of the spring board from a height of 6 inches if the modulus of elasticity is 1,200,000 pounds per square inch and the effect of the mass of the spring is neglected.

Since a variable load is usually applied somewhat gradually, the stress produced by a live load is generally somewhat smaller than twice the stress of an equal static load. The mass of the body which is subjected to the load and the natural vibration period of the parts of this body are factors which modify the maximum stress.

When a locomotive runs over a bridge, the effective unit stress may be 50 per cent. greater than that caused by its weight alone.

A structural engineer would say that an impact factor of 50 per cent. should be added to the live load stress.

For an impact formula for bridges, see American Railway Engineering Association *General Specifications for Steel Railway Bridges for Fixed Spans Less than* 300 *Feet in Length*, Second Edition, 1923, page 7.

For experiments on the impact of moving trains, see paper by F. E. Turneaure, *Transactions* of *American Society of Civil Engineers*, vol. XLI, pages 410–466.

169. Proof of the Proposition of Equivalent Deflections.— In Article 75, the proposition of equivalent deflections was given but not proved. This proposition is: *If A and B are two points of an elastic body, the deflection at A which is caused by a given load at B is equal to the deflection at B which is caused by the same load at A.*

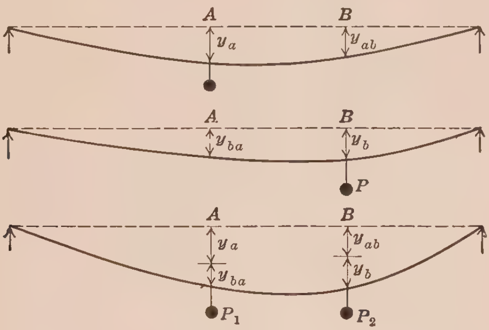


FIG. 187.

Figure 187 shows a beam which is supported at the ends. When a load P is placed at A, the deflection under the load is y_a and the deflection at B, which is caused by this load, is y_{ab}. In the second part of Fig. 187, the load P has been placed at B. The deflection under the load is y_b and the deflection at A is y_{ba}. The proposition of equivalent deflections states that $y_{ab} = y_{ba}$. This will now be proved: Let the load P_1 be placed on the beam at A and an equal load P_2 be then placed at B. The work done at A when the load P_1 is applied is $\frac{P_1 y_a}{2}$. The point B is lowered a distance y_{ab} when the load P_1 is applied at A but, since there is no force at B, no work is done. Let the load P_2 be now applied at B. The deflection at B is y_b and the work is $\frac{P_2 y_b}{2}$. At the same time the point A is deflected an additional y_{ba}

under the full load of P_1, and the additional work is P_1y_{ba}. The total work is

$$\frac{P_1y_a}{2} + P_1y_{ba} + \frac{P_2y_b}{2}. \tag{1}$$

If the load P_2 be placed on B first and then the load P_1 on A,

$$\text{work} = \frac{P_2y_b}{2} + P_2y_{ab} + \frac{P_1y_a}{2}. \tag{2}$$

Since the work represented by Equation (1) is equal to the work represented by Equation (2), the two expressions are equal. When P_1 is equal to P_2 the first and last terms of Equation (1) are equal to the first and last terms of Equation (2), hence

$$Py_{ba} = Py_{ab}; \quad y_{ba} = y_{ab}.$$

While the proposition has been proved by a beam which is supported at the ends, it applies equally well to any elastic solid with any number of supports.

Professor G. E. Beggs has applied this principle to models of arches and many other types of indeterminate structures. See paper entitled "An Accurate Mechanical Solution of Statically Indeterminate Structures by the Use of Paper Models and Special Gages," which was presented to The American Concrete Institute, Feb., 1922.

CHAPTER XVII

THEORIES OF ELASTIC LIMIT AND FAILURE

170. Principles Involved.—In Chapter XV, methods are given for finding the maximum tensile, compressive, and shearing stresses developed by a combination of stresses. It is a question which of these stresses determines the elastic limit and the failure. Also in Chapter I it was shown that stress in one direction causes a deformation in the opposite sense in all directions at right angles to the direct applied force. For instance, if there is a compressive stress along the X axis producing unit deformation δ, there is unit elongation $\sigma\delta$ along the Y and Z axes. If, at the same time, there is a tensile stress along the Y axis, the total elongation along that axis is that due to the tension in its direction in addition to the elongation due to the compression along the X axis. It is a question whether the failure which may occur at right angles to the tensile stress is influenced in any way by the additional elongation due to the compression.

As a result of these various considerations there are several theories to account for the relation of the stresses to the elastic limit and the failure.

171. The Maximum Stress Theory.—The *maximum stress theory*, called also *Rankine's* theory, assumes that failure is due to the single stress which is the largest, without reference to other stresses in other directions, except insofar as the components of these stresses affect the value of the maximum unit stress. If a block is subjected to a tensile stress s_t and to another tensile stress s_v at right angles to the first stress, if s_t is greater than s_v, the maximum stress is equal to s_t. This may be shown by resolution as in Article 155 or by means of Equation (11) of that article when s_s is equal to zero. According to the maximum-stress theory, the block will fail by rupture along a plane approximately normal to the maximum stress when this stress s_t reaches the ultimate strength of the material and the value of s_t to produce rupture is independent of the other stress s_v.

While this theory can hardly be said to be accepted by any one who seriously considers the subject. it is, nevertheless consider-

ably used in practice. In a boiler, for instance, the circumferential tensile stress tending to rupture the shell longitudinally is twice as great as the longitudinal tensile stress, and it is customary to calculate from the circumferential stress alone without reference to the other.

172. The Maximum Strain Theory.—This theory, which is also called *St. Venant's* theory, assumes that a solid reaches its elastic limit when the unit deformation reaches a given limit and that there is an ultimate unit deformation which cannot be exceeded without rupture, no matter in what way the stresses are applied which cause the deformation.

Suppose a block, Fig. 188, is subjected to a direct tensile stress of s_t and to a compressive stress at right angles thereto of s_c, and suppose the material reaches its elastic limit in tension when the unit elongation is 0.001. According to the maximum strain theory, if the unit elongation which is due to tension is 0.0008 and there is an additional unit elongation of 0.0002 in the same direction which is due to transverse compression, the combined unit elongation of 0.001 brings the material to the elastic limit.

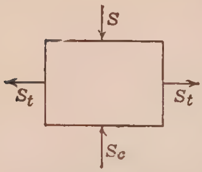


FIG. 188.—Tension and compression of right angles.

The tensile strength of some materials is much smaller than the compressive strength. If the ratio of the tensile strength to the compressive strength is less than Poisson's ratio for the material, a compressive load should cause failure by transverse tension. This is what seems to happen with porcelain and concrete. A porcelain rod, 1 inch in diameter and 16 inches long, supported a compressive load of 20,000 pounds per square inch and failed by splitting lengthwise. When porcelain is tested in tension, the heads of the specimen must be much larger than the minimum section, or the specimen will fail at the grips. Figure 189 shows the form of a series of bars of rectangular section. The pressure was transmitted to the heads from the grips through leather or lead sheets. Instead of failing at the minimum section, the bars failed along a curved surface AB at one of the heads. According to the maximum strain theory, the unit deformation across this curved surface which was caused by tension and the unit deformation which was caused by compression

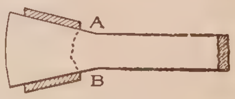


FIG. 189.

were together greater than the unit deformation in the smaller sections which was caused by tension alone.

The behaviour of porcelain under stress strongly supports the maximum strain theory for *brittle materials.*

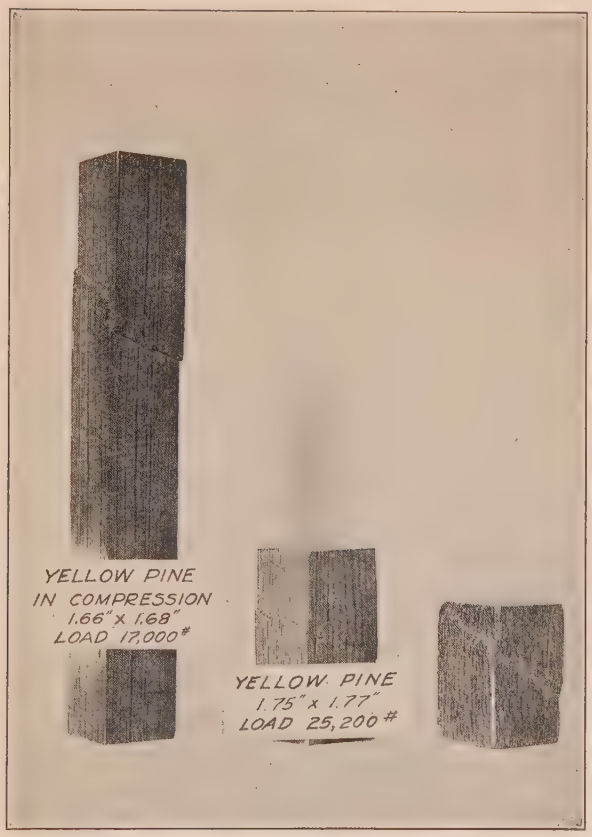


FIG. 190.—Timber in compression.

173. The Maximum Shear Theory.—This is frequently called the Guest* theory or the Guest-Hancock† theory. According to this theory, a given material reaches its elastic limit in tension

* See J. J. GUEST, "On the Strength of Ductile Materials under Combined Stress," *Philosophical Magazine*, July, 1900, pages 69–132.

† E. L. HANCOCK, "The Effect of Combined Stress on the Elastic Properties of Steel," *Proceedings of the American Society for Testing Materials*, 1905, pages 179–186; 1906, pages 295–307.

or compression, when the unit shearing stress, as calculated by Equation (12) of Article 155 or by Formula XXXI, reaches the elastic limit of the material in shear, and failure occurs when the unit shearing stress, as calculated by these formulas, reaches the ultimate shearing strength of the material.

As stated above there is no question as to the truth of the theory. Figure 190 shows three wooden blocks which were tested in compression. Failure has taken place by shear along planes at about 45 degrees with the direction of the stress, at which angle the unit shearing stress is a maximum. In a tensile test of soft steel the edges at the fracture are inclined at an angle of approximately 45 degrees to form the sides of the so-called crater.

It is evident that a bar in tension or compression will fail by shear provided it does not fail in some other way before the unit shearing stress (which, at 45 degrees, is one-half the unit tensile or compressive stress) reaches the ultimate shearing strength of the material. It is also evident that when the unit shearing stress reaches the elastic limit in shear, there will be large linear deformations which will appear as the elastic limit in tension or compression. The point upon which there is *not agreement* is whether a solid ever reaches its elastic limit in tension or compression before reaching the elastic limit in shear, and whether failure is always by shear.

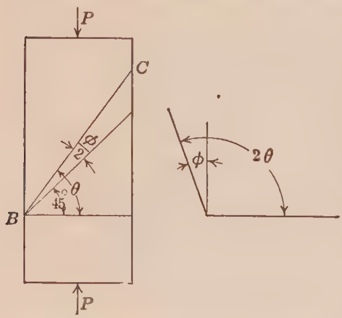


FIG. 191.—Shear failure caused by compression.

The tests made by Guest and Hancock were upon ductile materials, and neither of them claimed that the maximum shear theory applies to brittle solids.

The maximum unit shearing stress which is caused by tension or compression is at an angle of 45 degrees. Shear failure, however, frequently takes place at a much higher angle. The difference is due to friction. In Fig. 191, the component of the applied force, is $P \sin \theta$; the component normal to this plane is load P along the plane BC at an angle θ with the normal to $P \cos \theta$. If A is the area of the normal cross-section the area of the plane BC is $A \sec \theta$; and if s_s is the unit shearing strength, the shearing resistance is $s_sA \sec \theta$.

When a resolution is taken parallel to BC, the equation is

$$P \sin \theta = Pf \cos \theta + s_s A \sec \theta, \tag{1}$$

in which f is the coefficient of friction.

$$\frac{P}{s_s A} = \frac{\sec \theta}{\sin \theta - f \cos \theta}, \tag{2}$$

$$\frac{s_s A}{P} = \sin \theta \cos \theta - f \cos^2 \theta,$$

$$\frac{2s_s A}{P} = \sin 2\theta - f(1 + \cos 2\theta). \tag{3}$$

The load P is a minimum when the second member of Equation (3) is a maximum. When this is differentiated with respect to θ and equated to zero, the result is

$$\cos 2\theta + f \sin 2\theta = 0;$$

$$\cot 2\theta = -f; \ 2\theta = 90° + \text{arc} \tan f;$$

$$\theta = 45° + \frac{\text{arc} \tan f}{2}. \tag{4}$$

The angle whose tangent is the coefficient of friction is called the angle of friction, and is generally represented by the Greek letter ϕ. Equation (4) may be written

$$\theta = 45° + \frac{\phi}{2}. \tag{5}$$

Failure takes place along a plane which makes an angle of 45 degrees plus one-half the angle of friction with the plane normal to the compressive force.

174. Failure.—As previously stated, failure of ductile material in tension largely takes place by shearing at about 45 degrees with the direction of the tensile stress. In ductile steel test bars the failure is at about 45 degrees at the circumference of the rod, and is normal to the length at the middle. Some fairly soft steel fails by shear across the entire bar and does not form "craters." Non-ductile material, such as cast iron or porcelain fails at right angles to the direction of the tensile stress.

Since the shearing stress at 45 degrees is one-half the tensile stress, failure by shear indicates that the shearing strength is less than one-half the tensile strength.

Figure 192 shows a wooden bar which was tested in tension. To prevent failure at the grips, the section was cut down at

the middle. The fracture (not clearly shown in the print) is along lines at angles much greater than 45 degrees. Timber has small shearing strength parallel to the grain, which accounts for this kind of failure. In compresson, timber fails by shear at about 45 degrees, as shown in Fig. 190. The blocks shown in this figure are exceptional in that the shear occurs for a considerable distance in one plane. Generally the specimen shears for a short distance and then splits to another shear plane.

FIG. 192.—Timber in tension.

Shear failure in timber under compression is usually nearer to 45 degrees than to 45 degrees plus one-half the angle of friction. It seems that shearing displacement takes place by bending the fibers rather than by sliding.

Figure 193 shows pieces of wrought-iron pipe which have been tested in compression. The material flows under stress and finally splits.

Figure 194 shows hard brick in compression. The failure takes place at an angle much greater than 45 degrees. The coefficient of friction of brick on brick is relatively large, and the angle is somewhere near 45 degrees plus one-half the angle of friction.

Figure 195 shows two 4-inch by 4-inch blocks of 1:1 cement mortar, each of which failed by shearing at the ends and then splitting lengthwise. The longitudinal fracture may be explained as caused by the wedge action of the shear pyramids. Such wedge action would not, however, be likely to split the blocks the entire length. Another explanation is that this failure is due to the lateral expansion which results from the longitudinal pressure. If Poisson's ratio is 0.15, the compressive load of 4,000 pounds per square inch which was applied to these prisms would produce a lateral elongation equivalent to a tensile stress of 600 pounds per square inch. This easily accounts for the failure by the maximum strain theory. The friction of the compression heads of the testing machine resist the lateral expansion at the ends and prevents splitting throughout the entire length.

The bearing strength of a solid depends upon the relative size of the surface of contact and the entire dimensions of the body. In the treatment of bearing stress there are two limiting cases. The first is that shown in Fig. 196 in which the surface of contact

FIG. 193.—Metal in compression.

is equal to the entire cross-section of the body B, and the length in the direction of the applied force is at least equal to the thickness of the body. In this case the bearing strength is equivalent to the compressive strength. Used in this way, a soft material

FIG. 194.—Hard brick in compression.

like babbitt metal would show little bearing strength. Figure 197 shows a second case. Here the load is applied to a small portion of the body which is of unlimited extent or is confined laterally by another body. The portion outside of the loaded area acts as a hoop to prevent the lateral expansion. In this form, a

body composed of *separate particles* may have considerable bearing strength if the coefficient of friction is fairly high. Dry sand is an example. In a mass of wheat or flaxseed, where the coefficient of friction is small, the bearing strength is very low.

Fig. 195.—Cement in compression.

Figure 198 shows two types of bearing pressure intermediate between Figs. 196 and 197. When a post of wood or metal is placed on a concrete wall with no metal plate between them to transmit the pressure, the concrete wall or pier is given the form of one or the other support B of Fig. 198.

Cutting with a knife or chisel depends upon the bearing strength of the tool and of the material which is cut. The bearing strength of the tool under the conditions of Fig. 196 must be greater than the bearing strength of the material under

the conditions of Fig. 197. At first there is a depression in the material under the edge of the tool, as shown in Fig. 199, I. When the unit stress under the edge of the tool exceeds the bearing strength of the material, it is permanently pushed back.

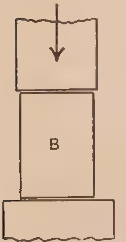


FIG. 196.—Bearing.

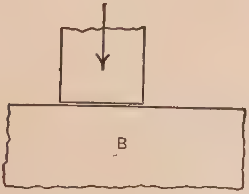


FIG. 197.—Bearing on large body.

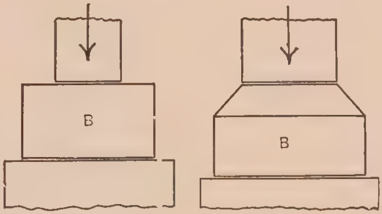


FIG. 198.—Cases of bearing pressure.

In a plastic non-porous material, some of the substance is forced up by the pressure, as shown in Fig. 199, II. In a porous body such as wood, there is an increase in density adjacent to the cutting surface. The wheel of a wagon cutting into soft earth illustrates both cases. If the earth is wet clay, it is pushed up at the sides of the tire. If it is dry loam, it is compressed under the tire.

I II III

FIG. 199.—Cutting.

When a cutting tool has penetrated a little distance, it acts as a wedge and exerts a tensile stress upon the material in front of its edge. This is shown in Fig. 199, III.

Figure 200 shows the behavior of a pair of scissors or shears. At the beginning, the cutting is due to the bearing stress on the cutting edges, as shown in Fig. 200, I. As the edges penetrate into the material, the bearing force is increased at each blade. These forces produce shearing stresses in all portions of the body in the plane of the cutting edges. The corresponding shearing deformations are shown by the dotted lines in Fig. 200, II. Fig. 201 represents the punching of a metal plate. The plate is bent a little at first, which makes the surface of contact a narrow ring at the edge of the punch and die. When the compressive stress on these rings exceeds the bearing strength of the plate,

cutting begins. This is followed by shear, as in the case of cutting with scissors.

Figure 202 shows some slugs which were punched from steel plates. The ends of these slugs are permanently curved. The sides show the appearance of the fracture. The two slugs

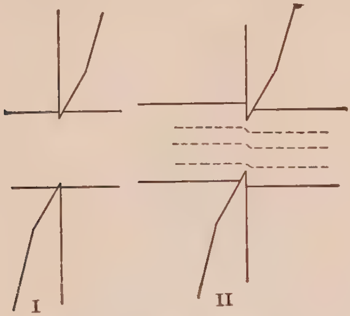


Fig. 200.—Cutting with shears.

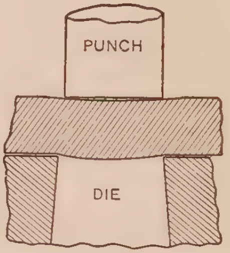


Fig. 201.—Punching a plate.

of least diameter were punched from a relatively thick plate. The bearing stress was too great, and the punch failed after making about a dozen holes.

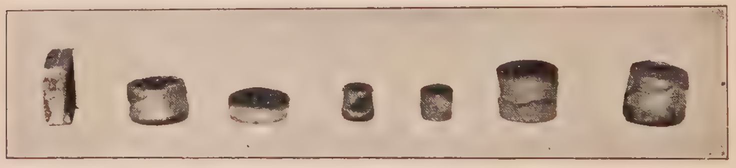


Fig. 202.—Slugs punched from steel plates.

175. Biaxial Loading.—The most common forms of biaxial loading consist of two tensions, one tension and one compression, or two compressions at right angles to each other. Figure 203 represents two tensile stresses of intensities s_t and s_v. With this form of loading, the maximum unit shearing stress is $\frac{s_t}{2}$, no matter how great s_v may be (provided it is not greater than s_t). If the maximum shear theory is valid, the material subjected to biaxial loading should reach its elastic limit at the same value of s_t irrespective of the direction or magnitude of s_v. On the other hand, a tensile stress s_v will diminish the unit strain in the direction of s_t and a compressive stress in the direction of

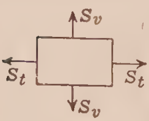


Fig. 203.

s_v will increase the unit strain in the direction of s_t. The unit strain in the direction of s_t is given by

$$\delta = \frac{s_t}{E} - \frac{s_v\sigma}{E}. \tag{1}$$

If the maximum-strain theory is the correct one, the unit stress in the direction of s_t at the elastic limit or yield point will be increased as s_v is increased, and will be diminished if s_v is changed to compression.

Prof. Albert J. Becker* has performed an extensive series of experiments with biaxial loading to determine these points.

These tests were made on hollow steel cylinders. The pressure of a liquid inside these cylinders produced a circumferential tensile stress, and an axial tensile stress. The axial stress was further increased by the direct pull of a testing machine. In each test the ratio of the circumferential unit stress to the axial unit stress was kept constant. Johnson's apparent elastic limit for the axial deformation was taken as the limit to be determined.

Table XVII gives approximately the results of one set of experiments.†

TABLE XVII.—BIAXIAL LOADING TESTS

Tube number	Ratio of circumferential stress to axial stress	At apparent elastic limit	
		Axial unit stress, lb./in.^{2}	Axial unit elongation
5	0.0	43,000	0.00165
1	0.240	46,000	0.00168
2	0.475	50,000	0.00152
4	0.69	50,000	0.00152
3	0.92	50,000	0.00140

Tube No. 5, with no circumferential stress, reached the elastic limit at the tensile stress of 43,000 pounds per square inch. The maximum unit shearing stress at 45 degrees was 21,500 pounds per square inch. If failure always takes place by shear, the other tubes should reach the elastic limit at the same unit stress. It

* A. J. BECKER, "The Strength and Stiffness of Steel under Biaxial Loading," *Bulletin* No. 85 of The University of Illinois Engineering Experiment Station.

† The data of Table XVII were estimated from the curves of Fig. 17 of *Bulletin* No. 85 of the University of Illinois Engineering Experiment Station.

is seen, however, that tube No. 1 reached the elastic limit at about 46,000 per square inch axial stress, and the others at about 50,000 per square inch. It will be noted that the axial unit deformation at the elastic limit is about the same for the first two tubes and is less for the other three. It is evident that there are two sets of limiting conditions which determine the elastic limit. The material reaches its elastic limit when the unit deformation is about 0.00166; it also reaches the elastic limit when the unit shearing stress becomes about 25,000 pounds per square inch. Tubes 5 and 1 reached the limiting deformation before reaching the limiting shearing stress. In the other tubes, the greater transverse tension so reduced the axial deformation that they reached the limiting unit shearing stress while the unit stain was still considerably below the limit.

From the entire series of tests, of which Table XVII is only a small part, Becker concludes:*

"For increasing values of the ratio of the biaxial stresses the yield-point strength follows the maximum-strain theory until the value of the shearing stress reaches the shearing yield point, then the shearing stress controls according to the maximum-shear theory. *There are thus two independent laws each dominant within proper limits instead of some single law as has heretofore been assumed.*"

Table XVII shows that the elastic limit in shear is about 60 per cent. of the elastic limit in tension. If the maximum shear theory were true for all cases, the elastic limit in shear would be 50 per cent. of the elastic limit in tension.

In thin cylinders, such as boilers, the longitudinal unit stress is one-half of the circumferential unit stress, and both are tension. If s_t is the unit circumferential stress, the unit longitudinal stress is $\frac{s_t}{2}$. The circumferential unit deformation is reduced by the longitudinal unit stress. If Poisson's ratio is ¼ the unit deformation circumferentially is

$$\delta = \frac{s_t - \frac{s_t}{8}}{E} = \frac{7s_t}{8E}, \tag{2}$$

and the actual unit deformation is only seven-eighths as great as that which would be produced by the circumferential unit stress acting alone. It is not customary to consider this in cal-

* *Bulletin* No. 85, Illinois University Engineering Experiment Station, page 85.

culating the strength of boilers. The error, when it is neglected, is on the side of safety. The error is really small, for the weak part of a boiler is at a joint. At a longitudinal joint, on account of the lapping of the plates or the butt straps, there is much more material to resist longitudinal tension than in the main plates, and the longitudinal unit stress is less than $\frac{s_t}{2}$. There is also an unequal distribution of both stresses on account of the material cut away to form the rivet holes. This causes an error on the side of danger. For these reasons, it is not advisable to make any allowance for the reduced strain which is due to the combination of stress.

176. Combined Tension and Shear.—It was shown in Article 151 that a combination of tensile stress with shearing stress parallel and perpendicular to it gives

$$\text{Max } s_t' = \frac{s_t}{2} + \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}; \text{ Formula XXXII.}$$

$$\text{Min } s_t' = \frac{s_t}{2} - \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2} \qquad (2)$$

The minimum is equivalent to a compressive stress

$$s_c' = -\frac{s_t}{2} + \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}. \qquad (3)$$

The unit deformation in the direction of the tensile stress is that produced by the maximum tensile stress plus the effect of the compressive stress at right angles to the tensile stress.

$$E\delta = \frac{s_t}{2} + \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2} + \sigma\left(-\frac{s_t}{2} + \sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}\right); \qquad (4)$$

$$E\delta = \frac{s_t}{2}(1 - \sigma) + (1 + \sigma)\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}. \qquad (5)$$

When Poisson's ratio is ¼

$$E\delta = \frac{3s_t}{8} + \frac{5}{4}\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}. \qquad (6)$$

It is well known that brittle materials, under these conditions, fail by tension. A cast-iron rod broken by torsion, or by torsion and bending combined, fails along a curve which is approximately normal to the maximum tensile stress.

The same is true of concrete. Figure 204 shows a characteristic failure of a reinforced-concrete beam supported at the ends and loaded at the third points. A diagonal crack starts near the bottom and runs up to the point of application of one load. Such cracks are usually found in reinforced-concrete beams which are

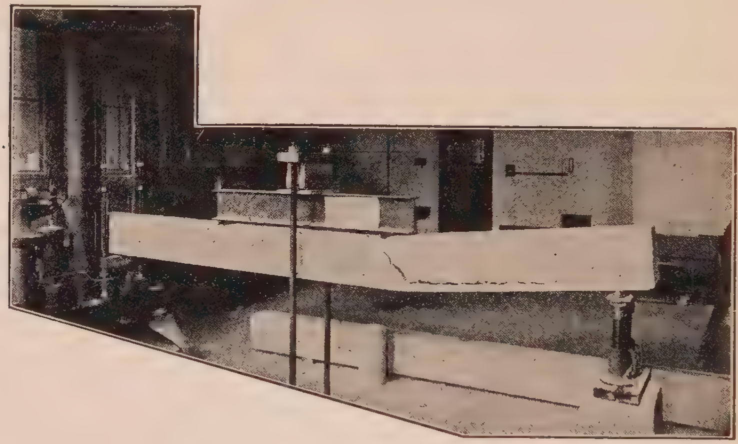


FIG. 204.—Failure of a reinforced-concrete beam.

so loaded as to develop large shear. Between the two concentrated loads on the beam of Fig. 204, the shear is zero (except that which is due to the weight of the beam) and the cracks in that part of the beam are vertical. The large crack in Fig. 204 extends horizontally along the line of the reinforcement. This crack, however, only opened as the bar approached total failure, while the diagonal crack is one of a number which appeared at about one-half of the ultimate load.

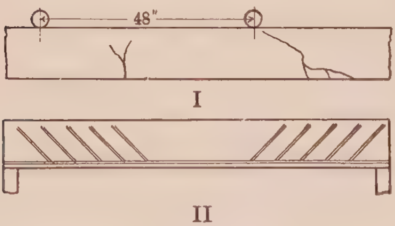


FIG. 205.

Figure 205, I is a drawing showing the same effect. The crack between the loads, running nearly vertical is called a tension crack, while the one on the right is called a shear crack. Both are really tension cracks, but the one on the right is due to the tension which results from the combination of tension with shear, while the one on the left is from tension alone.

To prevent failure by combined tension and shear, beams are reinforced as shown in Fig. 205, II. The diagonal bars are called shear bars and are placed in approximately the direction of the maximum resultant tensile stress and normal to the direction of the so-called shear cracks.

It is not known whether these failures are the results of maximum stress or maximum strain. Since there is not a great difference between the maximum stress (Formula XXXII) and the stress which corresponds with the maximum strain (Equations (5) or (6)), careful measurements will be required to differentiate between them. In calculations of this kind to find the allowable load, it is customary to use formula XXXII.

Combined tension and shear have been used to test the theories of failure as applied to ductile materials. J. J. Guest* tested cylinders of soft steel, iron, brass, and copper by combined tension and torsion. Some of these cylinders were solid but most of them were thin hollow tubes. The internal pressure of a liquid was also applied to some of the tubes. He determined the *yield point in tension* in the cylinder thus subjected to combined stress and came to the conclusion that the yield point is reached when the resultant shearing stress as calculated by Formula XXXI reaches a definite value. These researches, which are regarded as classic, are open to the criticism that several sets of determinations were made on the same tube, so that the elevation of the yield point when the material is stressed to its yield point becomes a disturbing factor.

Later, E. L. Hancock tested hollow steel cylinders under combined shear and tension. His experiments show definitely that, "The presence of a torsional stress lowers the unit stress and the unit strain at the elastic limit in tension and also lowers the modulus of elasticity, somewhat."†

Hancock calculated what he called the true tensile stress, by Equation (5) for Poisson's ratios of $\frac{1}{4}$ and $\frac{1}{3}$. He also calculated a value, which he called the true shearing stress, by the formula $(1 + \sigma)\sqrt{s_s^2 + \left(\frac{s_t}{2}\right)^2}$, and arrived at the conclusion that this "true unit shearing stress" determines the elastic limit of the material.

* *Philosophical Magazine*, July, 1900.

† *Proceedings of the American Society for Testing Materials*, 1906, page 301.

In the present state of our knowledge of combined shear and tension or compression it is best to calculate the unit shearing stress by Formula XXXI and the unit tensile stress by Formula XXXII and see that neither of these exceeds the allowable unit stress of its kind. In place of the tensile stress by Formula XXXII, the stress which corresponds with the maximum unit strain may be computed by Equation (5).

Problems

1. The allowable unit tensile stress is 15,000 and the allowable unit shearing stress is 10,000 pounds per square inch. Would a direct tensile stress of 12,000 pounds per square inch combined with a shearing stress of 7,000 pounds per square inch be allowable?
   Ans. No, since the maximum resultant tensile stress of 15,219 pounds per square inch exceeds the limit in tension.
2. If the unit tensile stress in Problem 1 were 8,000 pounds per square inch and the unit shearing stress were 9,000 pounds per square inch, what combined stress would determine the safety?
   Ans. The combined shearing stress, which is 9,849 pounds per square inch, is only a little below the limit.
3. If the allowable shearing stress is two-thirds of the allowable tensile stress, for what ratios of direct shear to direct tension will each govern the design?
   Ans. If s_s is less than $\frac{\sqrt{3}}{2}s_t$, the combined tensile stress governs the design.
   If greater, the combined shearing stress governs.

177. Elastic Hysteresis.—In an elastic body subjected to stress, the deformation lags behind the applied force. If a constant pull is applied to a body, the body will stretch quickly for a considerable amount and will continue to stretch slowly for some time. If the load is reduced, the body will shorten in a similar way. When a steel rod is stretched in an ordinary testing machine, the load is applied by means of the screws until the beam is balanced at the desired load. If the machine is then stopped, the beam slowly *falls* as the rod continues to elongate. This may continue for several minutes. If the beam is again lifted by running the machine for a short interval, it will fall more slowly the next time, if it comes down at all. If the stress is near the yield point, the beam will fall more quickly, and it may be necessary to lift it several times before the test bar will permanently support the load.

A similar effect is produced when the load is decreased. If the poise of the testing machine is set at a given load, and the tension in the test bar is reduced until it is balanced, the beam will rise slowly after the machine has stopped. This shows that the rod continues to shorten for some time.

TABLE XVIII.—TEST OF SOFT STEEL IN TENSION

Area of section, 0.600 square inch

Total load		Unit stress per square inch		Elongation when machine stopped	
When machine stopped	After one minute	When machine stopped	After one minute	In 8 inches	Unit
Pounds	Pounds	Pounds	Pounds	Inches	Inch
30	30	50	50	0.0	0.0
6,150	6,000	10,250	10,000	0.00270	0.00034
9,000	8,650	15,000	14,420	0.00405	0.00051
12,200	11,800	20,330	19,670	0.00550	0.00069
15,000	14,400	25,000	24,000	0.00675	0.00084
18,000	17,100	30,000	28,500	0.0082	0.00102
19,200	18,250	32,000	30,420	0.0085	0.00106
19,800	18,600	33,000	31,000	0.0087	0.00109
20,400	19,150	34,000	31,920	0.0092	0.00115
21,000	19,200	35,000	32,000	0.0121	0.00151
20,600	19,350	34,330	32,250	0.0450	0.00562
20,600	20,000	34,330	33,330	0.0530	0.00662
21,000	19,000	35,000	31,670	0.0733	0.00916
21,000	19,200	35,000	32,000	0.1740	0.02175
21,600	20,900	36,000	34,830	0.2121	0.02651
22,800	21,750	38,000	36,250	0.2393	0.02991
24,000	22,900	40,000	38,160	0.2773	0.03466
26,400	25,050	44,000	41,750	0.3873	0.04841
28,800	26,900	48,000	44,830	0.5506	0.06882
31,200	29,050	52,000	48,520	0.83	0.104
32,400	30,600	54,000	51,000	1.30	0.162
32,800	30,850	54,670	51,420	1.62	0.202
32,850	31,200	54,750	52,000	1.89	0.236
32,750	30,950	54,580	51,580	2.08	0.260
32,600	30,900	54,330	51,500	2.52	0.315
23,400	....	39,000	....	2.98	0.372

Broke at 23,400 pounds. The area of the neck was 0.196 square inch.

Table XVIII gives the results of a test of soft steel in tension. In making this test, the machine was run rapidly and then stopped. The reading of the load was then taken. After one minute, the poise was moved backward until the beam again balanced. For instance, the machine was balanced at 25,000 pounds per square inch. After one minute, the balance was at 24,000 pounds per square inch.

The curves of Fig. 206 were plotted from the results of Table XVIII. From these curves and the table, it is evident that speed

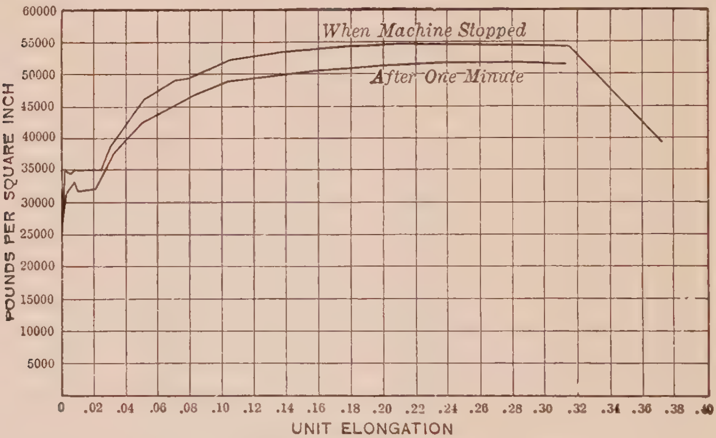


Fig. 206.—Effect of time on stress-strain diagram.

of the test is an important factor in determining the apparent strength and modulus of elasticity of a ductile material. When the machine was run rapidly, the apparent ultimate strength of this steel was 54,750 pounds per square inch. The quiescent load which it would permanently support is below 52,000 pounds per square inch.

If a ductile material is stretched nearly to the yield point, and the load then removed, a repetition of the load will show less lag in the deformation. There are usually internal strains in the test bar, which are relieved when the first pull is applied.

Table XIX gives part of the test of a rod taken from the same bar as that of Table XVIII. The tension was first raised to 42,000 pounds per square inch and then reduced to 50 pounds per square inch. There was a permanent set of 0.3 inch in the gage length of 8 inches. Curve I of Fig. 207 applies to this portion of the test up to the elastic limit. The remainder of this

curve has been omitted to save space. Table XIX gives the results of a second loading which followed this preliminary test. While the actual gage length at the beginning of this second run was 8.3 inches and the area was correspondingly smaller than 0.600 square inches, the unit stresses and unit deformations have been calculated from the original dimensions.

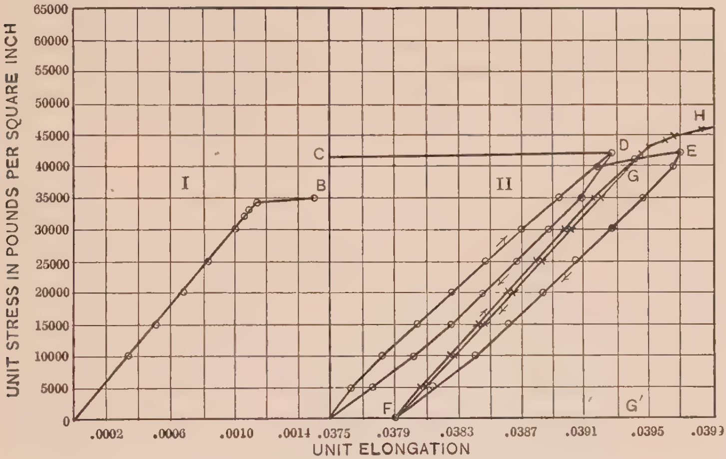


FIG. 207.—Stress-strain cycles.

It will be observed that the effect of time is much smaller than in Table XVIII. For instance, the unit stress falls from 35,000 pounds to 32,000 pounds in one minute in Table XVIII, while it falls from 35,000 to 34,830 pounds in Table XIX.

Table XIX shows also the increase of stress after a short interval when the load has been reduced. For instance, when the load was dropped from 25,000 pounds per square inch to 20,000 pounds per square inch and the machine then stopped, it increased to 20,375 pounds per square inch in the first minute.

Part II of Fig. 207 begins with the unit elongation of 0.0375. The portion of the preliminary test from *B* (of Part I) to *C* has been omitted. The curve was then drawn from *C* to *D*, which represents the maximum stress and elongation of the preliminary test. The load was then lowered to 50 pounds per square inch. Table XIX begins at this point at a unit elongation of 0.0375 inch, which corresponds with 0 of the table. From 5,000 to 40,000 pounds per square inch, the curve is nearly a straight line. At 40,000 pounds, there is a rapid change in slope and the

effect of time is very greatly increased. There is practically a new yield point at 40,000 pounds per square inch, whereas the original yield point was under 35,000 pounds per square inch. It will be seen from Table XVIII that when the apparent unit stress was 42,000 pounds per square inch it dropped to about

TABLE XIX.—REPEATED TEST OF SOFT STEEL

Total load		Unit stress per square inch		Elongation when machine stopped	
When machine stopped	After one minute	When machine stopped	After one minute	In 8 inches	Unit
Pounds	Pounds	Pounds	Pounds	Inch	Inch
30	30	50	50	0	0
3,000	3,000	5,000	5,000	0.00105	0.00013
6,000	6,000	10,000	10,000	0.00260	0.00032
9,000	8,900	15,000	14,830	0.00435	0.00054
12,000	11,900	20,000	19,830	0.00610	0.00076
15,000	14,950	25,000	24,920	0.00785	0.00098
18,000	17,825	30,000	29,710	0.00965	0.00121
21,000	20,900	35,000	34,830	0.01160	0.00145
24,000	23,600	40,000	39,330	0.01355	0.00169
24,600	24,200	41,000	40,330	0.01540	0.00192
25,200	24,550	42,000	40,920	0.01765	0.00221
24,000	23,900	40,000	39,830	0.01735	0.00217
21,000	21,150	35,000	35,250	0.01580	0.00197
18,000	18,100	30,000	30,170	0.01420	0.00177
15,000	15,075	25,000	25,125	0.01230	0.00154
12,000	12,225	20,000	20,375	0.01080	0.00135
9,000	9,300	15,000	15,500	0.00900	0.00112
6,000	6,225	10,000	10,375	0.00730	0.00091
3,000	3,125	5,000	5,210	0.00520	0.00065
30	120	50	200	0.00350	
30	30	50	50	0.00330	0.00041

40,000 pounds after one minute. The rod of Table XIX was raised to 42,000 pounds and then lowered and the new yield point on the second application of load is now raised to the load which it would *permanently* support with the elongation originally produced by 42,000 pounds per square inch.

When steel or wrought iron is stressed beyond the yield point, the yield point at the next application of load is found at the permanent stress previously reached.

The elastic limit is also raised. On the other hand, if the *yield point* in *tension* is *raised* by straining the bar beyond its original yield point, the *yield point* in *compression* is *lowered* so that the length of the *interval between these two yield points remains nearly constant.*

The descending curve from D is concave toward the left, while the ascending curves are concave toward the right. This difference in curvature always leaves an area between the ascending and descending curves, even when the stress is so low that

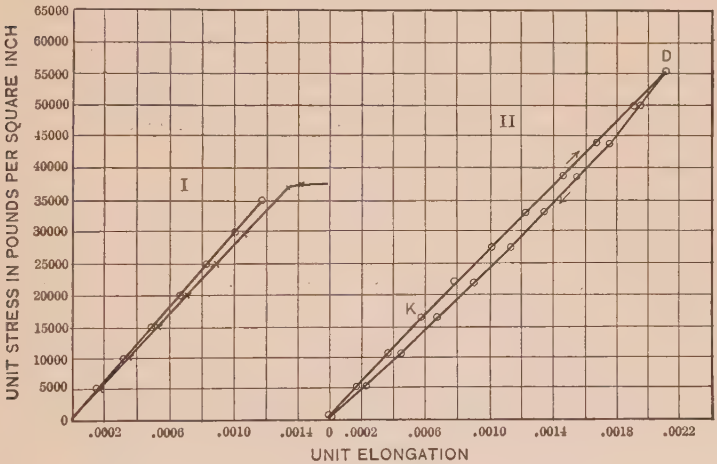


FIG. 208.—Watertown Arsenal test.

there is no permanent set. This area represents the energy which is lost in the cycle. The curves which enclose this area form the *loop* of *elastic hysteresis.*

At F, which represents the end of Table XIX, the bar was allowed to rest for 40 hours without load. The load was then raised to 40,000 pounds per square inch at G and again lowered to 50 pounds per square inch. It will be seen that the hysteresis loop has a small area, and that both the ascending and descending curves are nearly straight. Also, there is no temporary set.

Figure 208 shows some of the results of similar experiments at the Watertown Arsenal ("Tests of Metals," 1886, Part 2, pages 1571–1617). These tests were made on eye-bars about 25 feet long. The gage length was 260 inches which made it possible to measure the elongation with great relative accuracy.

Table XX represents the first part of the test. The inital load was 1,000 pounds per square inch. After each load the machine was reversed to the initial load and a reading taken for set. Under the heading "Unit elongation," the table gives the values obtained by subtracting the original reading at the initial load from the reading at the given load. Under the heading of "Net unit elongation," the table gives the values obtained by subtracting the *set at initial load following a given reading* from the elongation at that load. Curve I of Fig. 208 shows both elongations. The line of greatest slope drawn through the circles represents the net unit elongation. The values of the modulus of elasticity as obtained by the two methods are quite different.

TABLE XX.—WATERTOWN TESTS OF STEEL EYE-BAR

Total load	Unit stress per square inch	Elongation		Set under initial load	Net unit elongation	*E*
		In 260 inches	Unit			
Pounds	Pounds	Inches	Inch	Inches	Inch	
5,250	1,000	0.0	0.0			
26,250	5,000	0.0456	0.000175	0.0065	0.000150	26,670,000
52,500	10,000	0.0915	0.000352	0.0085	0.000319	28,210,000
78,750	15,000	0.1369	0.000526	0.0089	0.000485	28,860,000
105,000	20,000	0.1815	0.000698	0.0096	0.000661	28,750,000
131,250	25,000	0.2264	0.000871	0.0101	0.000832	28,850,000
157,500	30,000	0.2720	0.001046	0.0109	0.001004	28,880,000
183,750	35,000	0.3194	0.001229	0.0147	0.001172	29,010,000
196,000	37,330	0.3459	0.001330			
198,000	37,710	0.3700	0.001423			
200,000	38,090	0.5665	0.002179			
204,750	39,000	1.07	0.0041			
210,000	40,000	2.35	0.0090			
369,000	70,286	......	....	30.42		

After the load of Table XX was released, the bar was again tested. The results for one cycle are given in Table XX and by Curve II of Fig. 208. The unit elongation was calculated from the original length of 290.4 at the beginning of this test instead of the length of 260 inches at the beginning of the first run. Readings were taken at a given load and again after an

interval of three minutes. With this large gage length, the elongations could be measured with a high degree of accuracy. The results agree with those of Table XIX. With increasing loads, the unit elongation continued to increase. With decreasing

TABLE XXI.—WATERTOWN TEST OF STEEL EYE-EAR REPEATED

Area, 4.70 square inches. Gage length, 290.4 inches.
Bar previously stretched from 260 inches by a load of 369,000 pounds

Total load	Unit stress per square inch	Elongation in 290.4 inches		Immediate unit elongation
		Immediate	After three minutes	
Pounds	Pounds	Inch	Inch	Inch
5,250	1,117	0		
26,250	5,585	0.0503	0.0504	0.000173
52,500	11,170	0.1093	0.1099	0.000376
78,750	16,755	0.1685	0.1690	0.000582
105,000	22,340	0.2299	0.2309	0.000792
131,250	27,925	0.2922	0.2935	0.001006
157,500	33,510	0.3562	0.3575	0.001226
183,750	39,095	0.4209	0.4222	0.001449
210,000	44,680	0.4868	0.4885	0.001676
236,250	50,265	0.5533	0.5549	0.001905
262,500	55,850	0.6209	0.6230	0.002148
236,250	50,265	0.5660	0.5659	0.001949
210,000	44,680	0.5082	0.5080	0.001750
183,750	39,095	0.4491	0.4489	0.001546
157,500	33,510	0.3886	0.3880	0.001338
131,250	27,925	0.3262	0.3252	0.001123
105,000	22,340	0.2631	0.2620	0.000906
78,750	16,755	0.1980	0.1971	0.000682
52,500	11,170	0.1331	0.1319	0.000455
26,250	5,585	0.0655	0.0627	0.000225
5,250	1,117	0.0048	0.0030	0.000016
	Bar rested 15 hours under initial load.			
5,250	......	−0.0054		

loads, it continued to decrease. The difference for a three-minute interval was never more than one-half of one per cent. of the total elongation.

These tests show that there is a relatively large hysteresis the first time the load is applied and much less hysteresis when the loading is repeated. There is also some set, which slowly vanishes.

The form of the ascending curve at the second application of the load depends largely upon the length of time which has elapsed after the first load was removed. If the second loading is applied before most of the temporary set has vanished, the curve will be very steep at first and will have an apparent elastic limit at a low stress. This is seen in Fig. 207 where there is a change in slope at 5,000 pounds per square inch, and then practically a straight line up to D. The curve starting from F has no such bend, because the bar rested before this load was applied.

178. Fatigue of Metals.—There is a considerable area between the ascending and descending curves of a "loop" in Figs. 207 and 208. Since one coördinate is force and the other is displacement, this area represents work which is expended in stretching the bar and is not returned when the load is released.

Since energy is lost in a cycle of this kind, it is natural to expect that a great number of repetitions of stress would cause failure at a maximum stress lower than the ultimate strength of the material. The experiments of Wöhler and many other investigators show that this is true. Steel which is subjected to a complete reversal of stress will fail under a great number of repeated applications of a stress which is considerably below the proportional elastic limit. For instance, steel of 0.49 of one per cent. carbon which was tested by Moore and Jasper* had an ultimate tensile strength of 91,500 pounds per square inch and a proportional elastic limit of 44,700 pounds per square inch. When this steel was tested under complete reversal of stress, the *endurance limit* was found to be 33,000 pounds per square inch. If the unit stress was changed from 33,000 pounds per square inch tension to 33,000 pounds per square inch compression the specimen would stand an indefinite number of repetitions without failure. Two specimens of this material were each subjected to over 100,000,000 repetitions of a stress which varied from 33,100 pounds per square inch tension to 33,100 pounds per square inch compression. Neither of these failed. A third

* *Bulletin No. 136* of the Engineering Experiment Station of the University of Illinois, by H. F. Moore and T. M. Jasper, pages 23 and 31.

specimen was subjected to one billion repetitions of a stress which varied from 33,000 pounds per square inch tension to the same compression without failure. The stress of 33,000 pounds per square inch is, therefore given as the *endurance limit* of this steel for complete reversal of stress. When the stress in this material varied from 34,000 to −34,000 pounds per square inch, each of the two specimens tested failed at about 4,000,000 repetitions. At 37,000 pounds per square inch, the failure was at 1,225,000 repetitions; and at 50,000 pounds per square inch, the failure was at 42,000 repetitions.

If the stress is not completely reversed, the endurance limit is higher. The smaller the range between the maximum and minimum stress the greater is the maximum stress which the material will endure for an indefinite time. The carbon steel above mentioned* has an endurance limit of 33,000 pounds per square inch for complete reversal of stress. The same material stood an indefinite number of repetitions when the stress ranged from 36,000 pounds tension to 21,600 pounds compression, from 47,000 pounds tension to zero, from 60,000 pounds tension to 12,000 pounds tension, or from 69,000 pounds tension to 34,500 pounds tension. These tests and many others show that the endurance is higher the smaller the range of stress.

179. Design for Variable Stress.—A number of methods have been proposed for designing members which are subjected to variable loads. This may be done by reducing the allowable unit stress or by adding an increment to the applied load.

Professor John Goodman† suggested a rule for this purpose. Goodman's "dynamic" rule is: *Add to the maximum load the difference between the maximum and minimum loads and treat the sum as a static load.* Under this rule, a load which varied from 12,000 pounds tension to 12,000 pounds compression would be equivalent to $12{,}000 + 24{,}000 = 36{,}000$ pounds tension or compression; a load which varied from 12,000 pounds tension to 6,000 pounds compression would be equivalent to $12{,}000 + 18{,}000 = 30{,}000$ pounds tension; and a load which varied from 12,000 pounds tension to 4,000 pounds tension would be equivalent to 20,000 pounds tension.

* *Bulletin No. 136* of the Engineering Experiment Station of the University of Illinois, page 77.

† Goodman's "Mechanics Applied to Engineering," page 535.

Problems

1. If the maximum allowable unit stress for a given material is 15,000 pounds per square inch, what is the required area of cross-section when the applied load varies from 20,000 pounds to 30,000 pounds?
Ans. 2.67 square inches.

2. What is the cross-section required to carry a load which varies from 30,000 pounds compression to 60,000 pounds tension if the allowable static unit stress is 12,000 pounds per square inch?
Ans. 12.5 square inches.

Moore and Jasper have shown* that Goodman's "dynamic" errs on the side of safety. They suggest the following formula, which is the equation of a straight line.

$$S_r = S_{-1}\left(\frac{r+3}{2}\right),$$

in which r is the ratio of the minimum stress to the maximum stress, S_r is the endurance limit for the ratio r, S_{-1} is the endurance limit for complete reversal. When the stress is reversed, r is negative. For instance, if the stress changes from 8,000 pounds per square inch compression to 20,000 pounds per square inch tension, r is -0.4. To use this formula, of course, the endurance limit for complete reversal must be determined experimentally. Moore and Jasper state further: "This formula can be used only up to the limit at which the maximum unit stress set up reaches the proportional elastic limit of the material, and for most steels this eliminates ratios of minimum stress to maximum stress greater than zero. Beyond the proportional elastic limit the static properties of the steel are the governing factors rather than the fatigue properties."

Problems

3. A load varies from 20,000 pounds compression to 40,000 pounds tension If the endurance limit for complete reversal is 32,000 pounds per square inch, what is it for this loading?
Ans. $S_{-.5} = 32{,}000 \times 1.25 = 40{,}000$ pounds per square inch.

4. The endurance limit for complete reversal for 3.50 nickel steel tested at the University of Illinois was 60,000 pounds per square inch. What should it be when the load changed from 100,000 pounds tension to 40,000 pounds compression.
Ans. 78,000 pounds per square inch. One set of tests gave 83,000 and another gave 81,000 pounds per square inch.

* *Bulletin No. 136* of the Engineering Experiment Station of the University of Illinois, pages 82–89.

180. Determination of Endurance Limit by Rise of Temperature.—The determination of the endurance limit by subjecting each specimen to a variable load for several million times is a long drawn out process. Results which agree very closely with these long fatigue experiments may be secured in a short time. When work is expended on a body, heat is generated. When a specimen of steel is subjected to repeated stresses and the rise in temperature is measured, it has been found that there is a sudden increase in rate of generation of heat, and, consequently, in the rate of rise in temperature, when the endurance limit is reached. This principle, which has been applied successfully by Putnam and Harsch, makes it possible to find the endurance limit of a test specimen in a relatively short time.*

181. Crystallization under Repeated Stress.—When steel fails under repeated applications of load, the fracture has a crystalline appearance. For this reason it was long thought that repeated stresses cause the formation of crystals in the steel. Microscopic† examination shows that all steel is crystalline and that crystals do not form at atmospheric temperatures. The crystalline appearance of the fracture is due to the fact that the fracture has taken place across the crystals of the steel.

* W. J. Putnam and J. W. Harsch, Bulletin No. 124 of the Engineering Experiment Station of the University of Illinois, pages 119–127.

† For a complete list of papers on the Fatigue of Metals see Bulletin No. 124 of the Engineering Experiment Station of The University of Illinois, pages 168–178.

CHAPTER XVIII

CURVED BEAMS AND HOOKS

182. Stresses in Curved Beams.—Figure 209 represents a portion of a curved beam between two planes AB and CD, each of which passes through the center of curvature of the beam. The plane AB, at the left end, is regarded as fixed, while the plane CD, at the right end, is rotated through an angle θ to the position $C'D'$ when the beam is bent. The unit stresses in a beam of this kind *do not vary* directly as the distance from the neutral axis, because

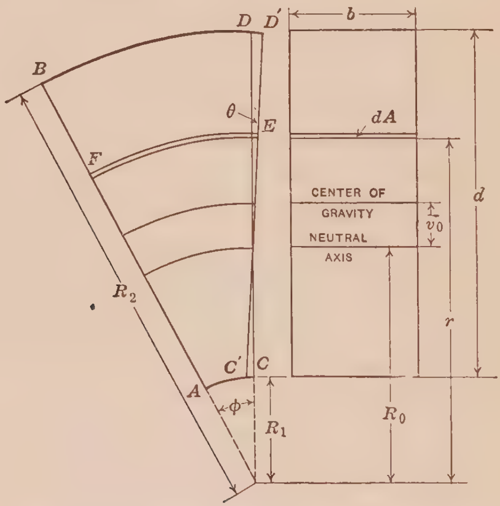


Fig. 209.—Curved beam of rectangular section.

the length of the filaments are not the same. If the neutral axis were midway between C and D the elongation DD' would be equal to the compression CC' but the *unit elongation* at the top would be less than the *unit compression* at the bottom, since the original length BD is greater than AC.

Since the length of any filament, such as EF is proportional to its distance from the center of curvature, the unit deformation and the unit stress vary as the quotient of the distance from the neutral axis divided by the distance from the center of curvature.

In Fig. 209, R_1 is the inside radius, R_2 is the outside radius, R_0 is the radius of the neutral surface, r is the radius of any filament and v_0 is the distance of the neutral axis from the center of gravity of the cross-section. The angle at the center of curvature subtended by the portion of the beam is ϕ, so that the original length of any filament is $r\phi$. The angle through which the plane CD is turned when the beam is bent is θ. If the assumption that a cross-section of a beam remains plane when the beam is bent *be valid for curved as well as for straight beams*, the deformation of a filament at a distance $r - R_0$ from the neutral axis is $(r - R_0)\theta$ and

$$\text{unit deformation} = \frac{(r - R_0)\theta}{r\phi}. \tag{1}$$

$$\text{unit stress} = s = \frac{E(r - R_0)\theta}{r\phi} = k\left(1 - \frac{R_0}{r}\right). \tag{2}$$

At the innermost fibers

$$S_1 = k\left(1 - \frac{R_0}{R_1}\right). \tag{3}$$

At the extreme outer fibers

$$S_2 = k\left(1 - \frac{R_0}{R_2}\right). \tag{4}$$

The location of the neutral axis is found by means of the condition that the total stress across any section is zero.

$$\text{Stress on element of area } dA = k\left(1 - \frac{R_0}{r}\right)dA; \tag{5}$$

$$\text{total stress} = k\int_{R_1}^{R_2}\left(1 - \frac{R_0}{r}\right)dA = 0. \tag{6}$$

$$A = R_0\int_{R_1}^{R_2}\frac{dA}{r};\ R_0 = \frac{A}{\displaystyle\int^{R_2}\frac{dA}{r}}. \tag{7}$$

The resisting moment of the stress on the area dA is the product of this stress multiplied by distance $r - R_0$.

$$M = k\int_{R_1}^{R_2}\frac{(r - R_0)^2}{r}dA. \tag{8}$$

The values given by Equations (7) and (8) depend upon the form of the section, that is, upon the value of dA as a function of r.

To find the unit stress at any point in terms of the moment, eliminate k between Equations (2) and (8), and to find S at the

inner or outer fibers, use R_1 or R_2 for r in the equation thus obtained.

183. Curved Beams of Rectangular Section.—For a rectangular beam of unit width, $dA = dr$ and $A = R_2 - R_1 = d$, in which d is the depth. From Equation (2) of Article 182,

$$R_0 = \frac{A}{\int_{R_1}^{R_2}\frac{dr}{r}} = \frac{A}{\log\frac{R_2}{R_1}} = \frac{d}{\log\frac{R_1+d}{R_1}}. \tag{1}$$

$$v_0 = \frac{R_2+R_1}{2} - R_0 = R_1 + \frac{d}{2} - R_0. \tag{2}$$

Example

In a beam of rectangular section the inner radius is 4 inches and the outer radius is 8 inches. Find the distance of the neutral axis from the center of gravity of the section.

$$d = 4 \text{ inches}; \quad \frac{R_2}{R_1} = 2;$$

$$R_o = \frac{4}{\log_e 2} = \frac{4}{0.69315} = 5.771 \text{ inches.}$$

$$v_0 = 6 - 5.771 = 0.229$$

$$\frac{v_0}{d} = 0.0572.$$

In a rectangular beam of depth equal to the radius of the inner surface, the neutral axis is shifted toward the center of curvature 0.0572 of the depth, or nearly 6 per cent.

To find the resisting moment, substitute dr for dA in Equation (8) of the preceding article.

$$M = k\int_{R_1}^{R_2}\left(r - 2R_0 + \frac{R_0^2}{r}\right)dr = k\left[\frac{r^2}{2} - 2R_0r + R_0^2\log r\right]_{R_1}^{R_2} \tag{3}$$

$$M = k\left(\frac{R_2^2 - R_1^2}{2} - 2R_0(R_2 - R_1) + R_0^2\log\frac{R_2}{R}\right). \tag{4}$$

Substituting the value of R_0 from Equation (1),

$$M = k\left(\frac{R_2^2 - R_1^2}{2} - \frac{2(R_2-R_1)^2}{\log\frac{R_2}{R_1}} + \frac{(R_2-R_1)^2}{\log\frac{R_2}{R_1}}\right); \tag{5}$$

$$M = k\left(\frac{R_2^2 - R_1^2}{2} - \frac{(R_2-R_1)^2}{\log\frac{R_2}{R_1}}\right) = kd\left(\frac{R_2+R_1}{2} - \frac{d}{\log\frac{R_2}{R_1}}\right); \tag{6}$$

$$M = kd\left(\frac{R_2+R_1}{2} - R_0\right) = kv_0d. \tag{7}$$

For a rectangular section of width b instead of unity, multiply by b,

$$M = kv_0bd. \tag{8}$$

Equation (7) may be derived more quickly by taking moments with respect to an axis through the center of curvature. The moment arm of the stress on an element dA is now equal to r, and

$$M = k\int\left(1 - \frac{R_0}{r}\right)rdA = k\int(r - R_0)dr,$$

for a rectangular section of unit breadth.

$$M = k\left[\frac{r^2}{2} - R_0r\right]_{R_1}^{R_2} = k\left(\frac{R_2^2 - R_1^2}{2} - R_0(R_2 - R_1)\right);$$

$$M = kd\left(\frac{R_2 + R_1}{2} - R_0\right) = kv_0d. \tag{7}$$

For a beam of breadth b, Equation (7) becomes

$$M = kv_0bd \tag{8}$$

Eliminating k between (8) and Equation (2) of the preceding article,

$$S = \frac{M\left(1 - \frac{R_0}{r}\right)}{bv_0d} \tag{9}$$

In the above example, to find the unit stress at the inner fibers, where $r = R_1 = 4$ inches,

$$S_1 = \frac{M\left(1 - \frac{5.771}{4}\right)}{4 \times 0.229} = -\frac{0.4427M}{0.916} = -0.483M.$$

For a straight beam 1 inch wide and 4 inches deep, the unit stress in the extreme fibers is $0.375M$. The unit stress at the inner fibers of a rectangular beam for which the outer radius is twice the inner radius is $0.483 \div 0.375 = 1.288$ times as great as the stress in the extreme fibers of a straight beam of the same section.

Table XXII below gives the displacement of the neutral surface and the ratio of the unit stresses in the extreme inner concave surface and the extreme outer convex surface of a curved beam of rectangular section to the unit stresses in the extreme fibers of a straight beam of the same section.

TABLE XXII.—DISPLACEMENT OF NEUTRAL SURFACE AND EXTREME FIBER STRESSES IN CURVED BEAMS OF RECTANGULAR SECTION

Ratio of depth to inner radius, $\frac{d}{R_1}$	Distance of neutral axis from center in terms of depth, $\frac{v_0}{d}$	Unit stresses in extreme fibers compared with straight beams of same section	
		Concave	Convex
0.50	0.0326	1.153	0.875
1.00	0.0572	1.288	0.811
1.50	0.0753	1.409	0.764
2.00	0.0897	1.523	0.726
3.00	0.1120	1.733	0.682
4.00	0.1287	1.923	0.652

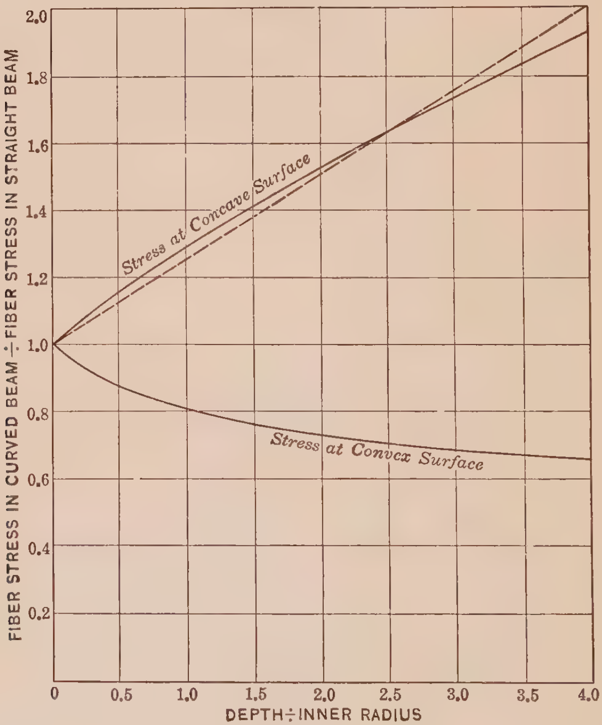


FIG. 210.—Stress at outer fibers of curved beam of rectangular section.

Figure 210 is plotted with $\frac{d}{R_1}$ as abscissa. The upper curve is the ratio of the unit stress at the concave surface of the curved beam to the unit stress in a straight beam of the same section.

The lower curve is a similar ratio for the unit stress at the convex surface of the curved beam.

Example

A rectangular beam, 4 inches wide, has an inside radius of 2 inches and an outside radius of 7 inches. By means of the curves of Fig. 210, find the maximum unit stress when the bending moment is 150,000 inch-pounds.

This moment would cause a unit stress of 9,000 pounds per square inch in a 4-inch by 5-inch straight beam. When the ratio of the depth to the inner radius is 2.5, the curve of Fig. 210 reads 1.63. The unit stress at the inner fibers is $9{,}000 \times 1.63 = 14{,}670$ pounds per square inch.

Problems

1. Verify Table XXII for $\frac{d}{R_1} = 3$.

2. A curved beam of rectangular section is 6 inches wide and 7 inches deep, and has a radius of 2 inches at the inner surface. Find the unit stress in the extreme fibers and the displacement of the neutral axis when the bending moment is 280,000 inch-pounds.

The stresses at the concave surface are numerically the greatest and are the most important, except in a curved beam of cast-iron which has the convex surface in tension.

The broken straight line of Fig. 210 differs little from the actual stress ratio for the concave surface. The slope of this line is 0.25 and its equation is

$$y = 1 + 0.25\frac{d}{R_1}, \tag{10}$$

in which y is the ratio of the unit stress in the curved beam at the concave surface to the unit stress in a straight beam of the same section. Based on this line, the unit stress at the concave surface of a beam is given by the *approximate formula*

$$S_1 = \frac{6M}{bd^2}\left(1 + 0.25\frac{d}{R_1}\right). \tag{11}$$

3. The inner radius of a curved beam of rectangular section is 5 inches, the outer radius is 13 inches, and the breadth is 4 inches. By means of Equation (11), find the unit stress at the concave surface when the bending moment is 25,600 inch-pounds.

Ans. $S = 600(1 + 0.4) = 840$ pounds per square inch.

184. Beams of T-section.—Figure 211 shows a T-section. The method of calculation is the same as for a rectangle, except that two sets of limits are required. Since the width of the flange is 4 inches and the thickness of the stem is 1 inch, the

two expressions must be multiplied by 4 and 1, respectively.

$$R_0 = \frac{A}{4 \log \frac{3}{2} + \log \frac{9}{3}} = \frac{10}{4 \times 0.40546 + 1.09861} = 3.676 \text{ inches}$$

Since the center of gravity is 2.6 inches from the inner surface,

$$v_0 = 2.6 - 1.676 = 0.924 \text{ inch.}$$

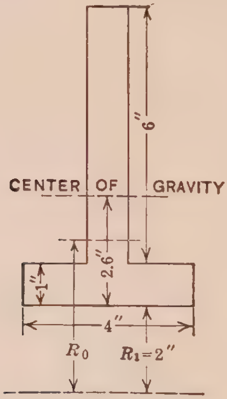


FIG. 211.—T-section.

The relative displacement of the neutral axis in a T-section or in an I-section is greater than in a rectangular section.

The resisting moment is calculated with respect to the axis of curvature.

$$M = 4k\int_2^3 (r - R_0)dr + k\int_3^9 (r - R_0)dr; \quad (1)$$

$$M = 4k\left[\frac{r^2}{2} - 3.676\right]_2^3 + k\left[\frac{r^2}{2} - 3.676\right]_3^9 = 9.24k; \quad (2)$$

$$k = \frac{M}{9.24}. \quad (3)$$

When this value of k is substituted in Equation (3) of Article 182,

$$S_1 = \frac{M}{9.24}\left(1 - \frac{3.676}{2}\right) = -0.0907M.$$

At the convex surface

$$S_2 = \frac{M}{9.24}\left(1 - \frac{3.676}{9}\right) = \frac{5.324M}{9 \times 9.24} = 0.064M.$$

Cast-iron beams are frequently made of T-section and used with the stem in compression in order to have the compressive stress

in the outer fibers greater than the tensile stress. In a straight beam of the section of Fig. 211, the stress at the top of the stem would be about 70 per cent greater than the stress at the bottom of the flange. With a curved beam the advantage of using a T-section is less. Unless the radius of curvature is large, the unit stress at the flange is greater than at the top of the stem.

Problem

Find the unit stress in the extreme fibers of a rectangular beam, which is 2 inches wide and 5 inches deep, if the inside radius is 2 inches. Compare with the results for the T-section above. If this rectangular beam is made of cast-iron, which surface should be in tension?

Ans. $S_1 = 0.212M$; $S_2 = 0.089M$.

185. Curved Beams of Circular Section.—From Fig. 212, the expressions for finding R_0 are

$$dA = -2\,a \sin\theta\, dr;$$
$$r = c + a\cos\theta;$$
$$dr = -a\sin\theta d\theta;$$
$$dA = 2a^2 \sin^2\theta d\theta, \tag{1}$$

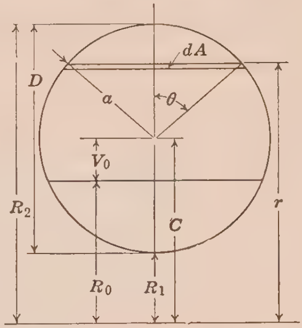


FIG. 212.—Circular section.

in which c is the radius from the center of curvature to the center of the circle, and a is the radius of the circle.

$$\frac{dA}{r} = \frac{2a^2 \sin^2\theta d\theta}{c + a\cos\theta}. \tag{2}$$

$$\frac{dA}{r} = \left(-2a\cos\theta + 2c - \frac{2(c^2 - a^2)}{c + a\cos\theta}\right)d\theta. \tag{3}$$

$$\int\frac{dA}{r} = \left[-2a\sin\theta + 2c\theta + 2\sqrt{c^2 - a^2}\left(\sin^{-1}\frac{a + c\cos\theta}{c + a\cos\theta}\right)\right]_0^\pi \tag{4}$$

$$= 2c\pi + 2\sqrt{c^2 - a^2}\left(\sin^{-1}\frac{a - c}{c - a} - \sin^{-1}\frac{a + c}{c + a}\right) \tag{5}$$

$$= 2c\pi - 2\pi\sqrt{c^2 - a^2} = 2\pi(c - \sqrt{c^2 - a^2}). \tag{6}$$

$$R_0 = \frac{\pi a^2}{2\pi(c - \sqrt{c^2 - a^2})} = \frac{c + \sqrt{c^2 - a^2}}{2}. \tag{7}$$

Since $c = \dfrac{R_2 + R_1}{2}$ and $a = \dfrac{R_2 - R_1}{2}$,

$$R_0 = \frac{R_1 + 2\sqrt{R_1R_2} + R_2}{4} = \frac{(\sqrt{R_1} + \sqrt{R_2})^2}{4}. \tag{8}$$

To find the resisting moment in a curved beam of circular section with respect to an *axis through the center of curvature,*

$$M = k\int\left(1 - \frac{R_0}{r}\right)r\,dA = k\int (r - R_0)a^2 \sin^2 \theta d\theta; \tag{9}$$

$$M = k\int\left((c - R_0)2a^2 \sin^2 \theta + 2a^3 \sin^2 \theta \cos \theta\right)d\theta; \tag{10}$$

$$M = ka^2\left[(c - R_0)\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{2}{3}a \sin^3 \theta\right]_0^\pi \tag{11}$$

$$M = ka^2\pi(c - R_0). \tag{12}$$

The moment in these equations is taken with respect to the axis of curvature. If the total tension is equal to the total compression, these forces form a couple and the moment about this axis (or any parallel axis) is the same as the moment about the neutral axis.

At the concave surface where $r = R_1$,

$$S_1 = k\left(1 - \frac{R_0}{R_1}\right).$$

When k is eliminated between this expression and Equation (12) the result is

$$S_1 = \frac{\left(1 - \frac{R_0}{R_1}\right)M}{\pi a^2(c - R_0)} \tag{13}$$

When the values of R_0 and c are substituted in Equation (13), the stress at the concave surface is found to be

$$S_1 = \frac{(3R_1 - 2\sqrt{R_1R_2} - R_2)M}{\pi R_1 a^2(R_1 - 2\sqrt{R_1R_2} + R_2)}. \tag{14}$$

At the convex surface where $r = R_2$,

$$S_2 = \frac{(3R_2 - 2\sqrt{R_1R_2} - R_1)M}{\pi R_2 a^2(R_1 - 2\sqrt{R_1R_2} + R_2)}. \tag{15}$$

Example

The inner radius is 4 inches and the outer radius is 9 inches. Find the displacement of the neutral axis from the center of gravity of the section, and find the ratio of the unit stress in the extreme fibers to the unit stress in a straight beam of the same cross-section.

$$R_0 = \frac{4 + 12 + 9}{4} = 6.25 \text{ inches.}$$

$$v_0 = 6.5 - 6.25 = 0.25 \text{ inch.}$$

The relative displacement, in terms of the radius is 0.25 ÷ 2.5 = 0.10. The relative displacement in this example is one-tenth of the radius of the section.

From Equation (14), the stress at the inner surface is

$$S_1 = \frac{(12 - 12 - 9)}{4\pi a^2(4 - 12 + 9)} = -\frac{9M}{4\pi a^2} = -\frac{2.25M}{\pi a^2}.$$

From Equation (15) the stress at the outer surface is

$$S_2 = \frac{(27 - 12 - 4)M}{9\pi a^2(4 - 12 + 9)} = \frac{11M}{9\pi a^2} = \frac{1.22M}{\pi a^2}.$$

In a straight beam, 5 inches in diameter,

$$S = \frac{M}{\frac{\pi a^3}{4}} = \frac{4M}{\frac{5\pi a^2}{2}} = \frac{1.6M}{\pi a^2},$$

Problems

1. A beam of circular section is 2 inches in diameter, and is curved so that the inner radius is 1 inch. Find the displacement of the neutral surface and the unit stress in the extreme fibers compared with the unit stress in a straight beam of the same section.

Ans. $v_0 = 0.134$ inch; $S_1 = 1.616S$; $S_2 = 0.705S$.

2. Show that in a beam of circular section the maximum possible displacement of the neutral axis is one-half the radius.

TABLE XXIII.—CURVED BEAMS OF CIRCULAR SECTION

Ratio of diameter to inner radius, $\frac{D}{R_1}$	Distance of neutral axis from center in terms of diameter $\frac{v_0}{D}$	Unit stresses compared with stresses in straight beam of same section: Concave, $\frac{S_1}{S}$	Convex, $\frac{S_2}{S}$
0.2	0.0114	1.071	0.935
0.4	0.0210	1.142	0.887
0.6	0.0293	1.207	0.841
0.8	0.0365	1.271	0.817
1.0	0.0429	1.332	0.791
1.5	0.0563	1.478	0.741
2.0	0.0670	1.616	0.705
2.5	0.0758	1.748	0.678
3.0	0.0833	1.875	0.656
4.0	0.0955	2.118	0.623

Figure 213 is plotted from Table XXIII. The relative stress at the concave surface does not differ greatly from that represented by the straight line,

$$S_1 = S\left(1 + 0.3\frac{D}{R_1}\right) = \frac{4M}{\pi a^3}\left(1 + 0.3\frac{D}{R_1}\right). \qquad (16)$$

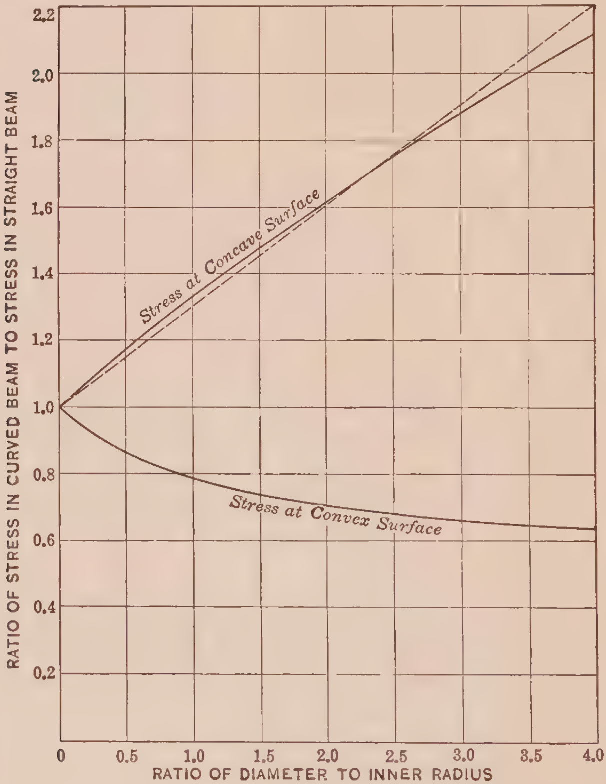


FIG. 213.—Unit stress in outer fibers of curved beam of circular section.

186. Curved Beams of Trapezoidal Section.—In a trapezoidal section, let C be the distance from the center of curvature to the point of intersection of the non-parallel edges of the section, and let m be the increase of width of the section per unit distance measured along the radius, Fig. 214. In the figure, C is greater than R_2 and m is negative. C may be less than R_1, in which case m is positive.

$$s = k\left(1 - \frac{R_0}{r}\right). \qquad (1)$$

An element of area is $m\ (r - C)dr$, and

$$\text{total stress} = km\int\left(1 - \frac{R_0}{r}\right)(r - C)dr; \qquad (2)$$

$$\text{total stress} = km\left[\frac{r^2}{2} - R_0r - Cr + CR_0 \log r\right]_{R_1}^{R_2}; \qquad (3)$$

total stress

$$= km\left(\frac{R_2^2 - R_1^2}{2} - R_0(R_2 - R_1) - C(R_2 - R_1) + CR \log \frac{R_2}{R_1}\right). \qquad (4)$$

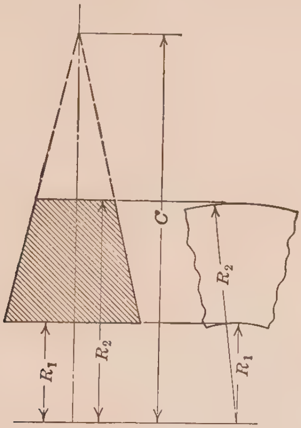


FIG. 214.—Trapezoidal curved beam.

If there is no resultant stress normal to the section, the total stress is zero, and

$$R_0 = \frac{\left(\frac{R_2 + R_1}{2} - C\right)d}{d - C \log_e \frac{R_2}{R_1}}, \qquad (5)$$

in which $d = R_2 - R_1$.

$$\text{When } C = 0,\ R_0 = \frac{R_2 + R_1}{2},$$

and the neutral axis is midway between the surfaces.

To find the moment of the trapezoidal section with respect to the center of curvature,

$$M = km\int(r - R_0)(r - C)dr; \qquad (6)$$

$$M = km\int\left(r^2 - (R + C)r + CR_0\right)dr; \qquad (7)$$

$$M = km\left[\frac{r^3}{3} - (R_0 + C)\frac{r^2}{2} + CR_0r\right]_{R_1}^{R_2};$$

$$M = kmd\left(\frac{R_2^2 + R_2R_1 + R_1^2}{3} - (R_0 + C)\frac{R_2 + R_1}{2} + CR_0\right). (8)$$

At the concave surface where $r = R_1$,

$$S_1 = \frac{(R_1 - R_0)M}{mR_1d\left(\frac{R_2^2 + R_2R_1 + R_1^2}{3} - (R_0 + C)\frac{R_2 + R_1}{2} + CR_0\right)}. \qquad (9)$$

At the convex surface where $r = R_2$, a similar formula gives the fiber stress,

$$S_2 = \frac{(R_2 - R_0)M}{mR_2d\left(\frac{R_2^2 + R_2R_1 + R_1^2}{3} - (R_0 + C)\frac{R_2 + R_1}{2} + CR_0\right)}. \qquad (10)$$

Example

A trapezoidal curved beam is 2 inches wide at the concave surface, 1 inch wide at the convex surface and 2 inches deep. The inner radius is 4 inches. Locate the neutral axis and find the maximum unit stresses in terms of the moment.

$R_1 = 4$ inches, $R_2 = 6$ inches, $C = 8$ inches, $d = 2$ inches, $m = -0.5$.

$$R_0 = \frac{(5 - 8)2}{2 - 8 \log_e \frac{3}{2}} = \frac{6}{1.2437} = 4.824 \text{ inches.}$$

The center of gravity of the section is 4.889 inches from the center of curvature and $v_0 = 0.065$ inch.

$$S_1 = \frac{(4 - 4.824)M}{0.5 \times 4 \times 2\left(\frac{36 + 24 + 16}{3} - 12.824 \times 5 + 8 \times 4.824\right)};$$

$$S_1 = \frac{0.824M}{4 \times 0.195} = 1.084M.$$

$$S_2 = \frac{(6 - 4.824)M}{6 \times 0.195} = 1.005M.$$

For a straight beam of the same section,

$$S_1 = \frac{12M}{13} = 0.923M;$$

$$S_2 = \frac{15M}{13} = 1.154M.$$

Equation (5) shows that the location of the neutral axis is independent of the slope. The unit stress at the extreme fibers is inversely proportional to m. The ratio of the unit stress of the curved beam to the unit stress of a straight beam of equal area of cross-section is independent of m.

For any given ratio of C to R_1, a set of values might be computed for the ratio of the maximum unit stress in the curved beam to the maximum unit stress in a straight beam of equal section, and a curve might be plotted as was done for beams of circular and rectangular sections. A set of such

curves might be made for different values of $\frac{C}{R_1}$. However, since the calculation of the center of gravity and section modulus of a trapezoidal section is as laborious as direct substitution in Equations (5), (9), and (10), it is not worth while to make these curves.

187. Hooks.—A hook is equivalent to a curved beam which is subjected to eccentric tension. As in a short block which is eccentrically loaded, the total pull P may be replaced by a pull P through the center of gravity of the section and a bending moment Pe, in which e is the distance of the center of gravity of the section from the line of the load. The direct tensile stress is $\frac{P}{A}$. If the hook is straight at the section under consideration, the tensile stress which is due to bending is $\frac{Pev}{I}$ and the total stress at the innermost fibers is

$$S_t = \frac{P}{A}\left(1 + \frac{ec}{r^2}\right). \tag{1}$$

At the outermost fibers, if the hook is straight at the section under consideration,

$$S = \frac{P}{A}\left(1 - \frac{ec}{r^2}\right). \tag{2}$$

Since the eccentricity of the load on a hook is always so great that $\frac{ec}{r^2}$ is greater than unity, S is a compressive stress.

While Equations (1) and (2) afford an *approximate* method of finding the maximum stresses in a hook or curved bar which is subjected to tension or compression, there is an error on the side of danger, unless the curvature at the section under consideration is relatively small. For accurate results, it is necessary to regard the hook as a curved beam in calculating the bending stress. At the innermost fibers of a hook,

$$S_t = \frac{P}{A} + S_1, \tag{3}$$

in which S_1 is the unit stress at the concave surface calculated for a curved beam with the moment Pe. At the outermost fibers,

$$S_c = S_2 - \frac{P}{A}, \tag{4}$$

in which S_2 is the bending stress at the convex surface calculated for a curved beam with moment Pe.

188. Curved Bars of Rectangular Section.—While hooks are not made with rectangular sections, curved bars frequently have this form.

Example

A curved bar of rectangular section is 2 inches wide. At the section farthest from the applied load the inner radius is 3 inches and the outer radius is 6 inches. The load is 3,000 pounds tension and passes through the center of curvature. Find the maximum unit tensile and compressive stress at the most remote section.

$R_1 = 3$ inches; $R_2 = 6$ inches; $d = 3$ inches; $b = 2$ inches.
$M = 3{,}000 \times 4.5 = 13{,}500$ inch-pounds; $A = 6$ square inches;
$\frac{P}{A} = 500$ lb./in.^{2}

$$R_0 = \frac{3}{\log_e 2} = \frac{3}{0.69315} = 4.328 \text{ inches.}$$

$$v_0 = 4.500 - 4.328 = 0.172 \text{ inch.}$$

The unit stress at the innermost fiber due to bending, by Equation (9) of Article 183 is,

$$S_1 = \frac{(3 - 4.328)13{,}500}{3 \times 2 \times 3 \times 0.172} = 5{,}790 \text{ lb./in.}^2 \text{ tension.}$$

$$S_t = 5{,}790 + 500 = 6{,}290 \text{ lb./in.}^2$$

At the convex surface,

$$S_2 = \frac{(6 - 4.328)13{,}500}{6 \times 3 \times 2 \times 0.172} = 3{,}645 \text{ lb./in.}^2 \text{ compression.}$$

$$S_c = 3{,}638 - 500 = 3{,}145 \text{ lb./in.}^2$$

If the bar were straight and the eccentricity were the same, the bending stress in the outer fibers would be

$$S = \frac{13{,}500 \times 6}{2 \times 3^2} = 4{,}500 \text{ pounds per square inch.}$$

The approximate value of the unit bending stress, as calculated by Equation (11) of Article183, is $4{,}500\ (1 + 0.25) = 5{,}625$ lb./in.^{2}, and the approximate S_t is $5{,}625 + 500 = 6{,}125$ lb./in.^{2}

Problems

1. A curved beam of square section is 2 inches wide. The inner radius is 3 inches and the outer radius is 5 inches. The load is 2,000 pounds and passes through the center of curvature. Find the maximum tensile and compressive stress. *Ans.* $S_t = 7{,}866$ lb./in.^{2}
2. A curved bar of rectangular section is 3 inches wide. The inner radius is 6 inches and the outer radius is 10 inches. The load is 6,000 pounds and the line of the load is 3 inches from the concave surface of the bar. Find the maximum tensile stress.

189. Hooks of Circular Section.—The problem of finding the unit tensile stress at the concave surface of a hook of circular

section is solved by means of Equation (14) of Article 185. If tension is regarded as positive, the complete expression for the unit stress at the concave surface is

$$S_t = \frac{Pe(R_2 + 2\sqrt{R_1R_2} - 3R_1)}{\pi a^2 R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} + \frac{P}{\pi a^2}; \tag{1}$$

$$S_t = \frac{P}{\pi a^2}\left(\frac{e(R_2 + 2\sqrt{R_1R_2} - 3R_1)}{R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} + 1\right). \tag{2}$$

At the convex surface, from Equation (15) of Article 185

$$S_c = \frac{P}{\pi a^2}\left(\frac{e(3R_2 - 2\sqrt{R_1R_2} - R_1)}{R_1(R_1 - 2\sqrt{R_1R_2} + R_2)} - 1\right). \tag{3}$$

Example

A hook of circular section is 2 inches in diameter. The inner radius of curvature is 3 inches and the outer radius is 5 inches. The load is 2,000 pounds with the line of its resultant 1 inch inside the concave surface. Find the unit stress in the extreme fibers.

$R_1 = 3$ inches; $R_2 = 5$ inches; $a = 1$ inch; $e = 2$ inches.

$$\frac{P}{\pi a^2} = 636.6 \text{ lb./in.}^2$$

$$S_t = 636.6\left(\frac{2(5 + 2\sqrt{15} - 9)}{3(3 - 2\sqrt{15} + 5)} + 1\right).$$

$$S_t = 636.6\left(\frac{2 \times 3.746}{3 \times 0.254} + 1\right) = 636.6 \times 10.83 = 6{,}894 \text{ lb./in.}^2$$

$$S_c = 636.6\left(\frac{2 \times 4.254}{5 \times 0.254} - 1\right) = 636.6 \times 5.699 = 3{,}628 \text{ lb./in.}^2$$

Problem

A hook of circular section is 3 inches in diameter. The inner radius of curvature is 4 inches and the distance from the center of the section to the load line is 3 inches. If the maximum allowable unit stress is 10,000 pounds per square inch, what is the safe load? *Ans.* 6,400 pounds.

190. Hooks of Trapezoidal Section.—Hooks are frequently made of trapezoidal section with the larger base toward the center of curvature. In the actual hooks the corners are rounded as in Fig. 215. Such a hook may be calculated as if it were the full trapezoidal section and the bending stress in the actual hook may then be computed by multiplying the stress obtained from the full trapezoid by the ratio of the moment of inertia of the full

trapezoid to the moment of inertia of the actual section. This moment of inertia may be calculated with respect to the center of gravity or with respect to the neutral axis of the full section.

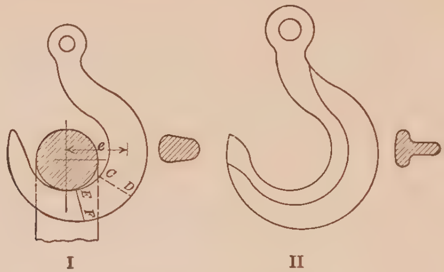


FIG. 215.—Hooks.

Example

A hook of trapezoidal section is 2 inches wide at the concave surface, 1 inch wide at the convex surface and 4 inches deep between these surfaces at the section most remote from the line of the load. The inner radius for this section is 4 inches and the line of the load is 2 inches from the concave surface. Find the unit stress in the extreme fibers when the load is 8,000 pounds.

To find the bending stress:

$$R_1 = 4 \text{ inches},\ R_2 = 8 \text{ inches},\ C = 12 \text{ inches},\ m = -\frac{1}{4}.$$

$$R_0 = \frac{24}{4.3178} = 5.558 \text{ inches.}$$

$$S_1 = \frac{(4 - 5.558)M}{4(37.333 - 105.348 + 66.696)} = \frac{1.558M}{5.276}.$$

The center of gravity is $1\frac{7}{9}$ inches from the concave surface so that the moment arm is $3\frac{7}{9}$ inches.

$$S_1 = 0.2952 \times 8{,}000 \times 3\tfrac{7}{9} = 8{,}920 \text{ lb./in.}^2$$

$$\frac{P}{A} = \frac{8{,}000}{6} = 1{,}333.$$

$$S_t = 10{,}253 \text{ lb./in.}^2$$

Problem

Solve the example above for the unit stress in the extreme outer fibers.

191. Variation of Dimensions and Curvature of Hooks.—In a hook or curved rod subjected to tension or compression parallel to its length, the sections most remote from the line of the load must have the greatest section. Since the unit stress at the concave surface is made smaller by increasing the radius of curvature it is desirable to make the radius of the more remote sections as

great as practicable. In Fig. 216 the section at A has the inner radius OA while the section at B has the smaller radius CB, and the section at E, where the moment is small, has a still smaller radius.

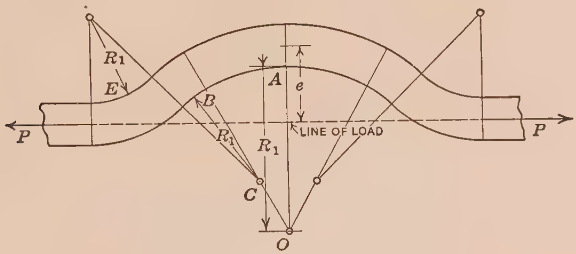


Fig. 216.—Curved bar.

The same principles may be applied to the design of a hook, but the increased length needed when the larger radius is used may require more material than would be necessary with the smaller radius and larger section.

In Fig. 215, I, the section CD is subjected to considerable shearing stress which must be taken into account in designing the section. The section EF is subjected, not only to shear, but also to a concentrated compressive stress for which due allowance must be made in the design.

CHAPTER XIX

CENTER OF GRAVITY

192. Center of Gravity.—When each of the particles which compose a body or system of bodies is subjected to a force which is proportional in magnitude to the mass of the particle and parallel to the similar forces in every other particle, the line of application of the resultant of these forces passes through the *center of gravity* of the body or system.

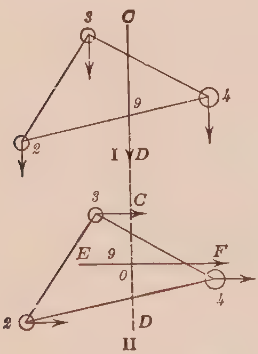


FIG. 217.—Center of gravity of three particles.

The location of the center of gravity is determined from the intersection of two such resultants.

Figure 217 represents three particles of relative masses 2, 3 and 4, united by weightless rods to form a single body. In Fig. 217, I, these particles are subjected to forces directed vertically downward. The resultant of these forces is a force of 9 units along the line CD. The center of gravity is located at some point on this line. In Fig. 217, II, the forces are horizontal, and their resultant is a horizontal force of 9 units along the line EF. The point O at the intersection of EF with CD is the center of gravity.

The center of gravity is also called the *center of mass*.

193. Determination of the Center of Gravity by Balancing.—The force with which the earth attracts the particles of a body is proportional* to the mass of each particle.

Since these forces are directed toward the center of the earth, they intersect at a distance of 4,000 miles, and may, therefore, be regarded as parallel within the limits of accuracy of measurement. The resultant force of gravity on any body passes through

* The difference in the force of the attraction of the earth which is due to inequality of distance from the center is about one part in ten million for a difference of one foot. This is negligible for ordinary bodies.

the center of gravity. A body may be held in equilibrium by a single force, provided that force is along the line of the resultant of all the other forces. When a body is supported by a flexible cord or by a point about which it is free to turn without friction, the center of gravity must be on the vertical line through the point of application of the cord or point (provided, of course, no forces are acting except gravity and the cord or point).

Figure 218 shows the same body as Fig. 217. In Fig. 218, I, the body is supported by a cord at C. A plumb line which is let fall from C passes through the center of gravity. In Fig. 218, II, the body is supported on a point or knife-edge at E and turns under the action of gravity until its center of gravity comes directly below the point of support. The intersection of the plumb line from E with the line CD (which has been marked in some way) gives the center of gravity O.

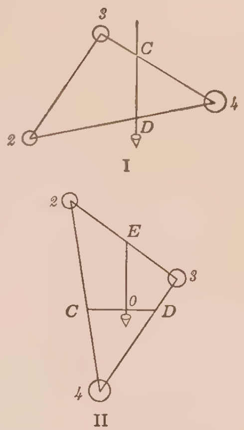


FIG. 218.—Location of center of gravity.

Since the center of gravity is usually surrounded by solid material, this method of finding it is of little practical use. It is sometimes valuable for relatively long bodies, especially if there are some plane surfaces which may be used as planes of reference. Figure 219 shows a beam balanced on a knife-edge. The center of gravity is in the vertical plane above the line of support. To completely locate the center of gravity of a solid, three planes must be determined which pass through it. The point of intersection of all three planes is the point required.

FIG. 219.—Center of gravity by balancing.

194. Center of Gravity by Moments.—In theoretical discussions, the center of gravity is usually located by moments. The plane of application of the resultant of any set of parallel forces may be determined by dividing the sum of the moments of these forces about any axis by the resultant force.

Example

Four masses are attached to a straight rod. These are: 16 pounds at the left end, 20 pounds at 7 feet, 8 pounds at 10 feet, and 12 pounds at 14 feet from the left end. The rod weighs 4 pounds and its center of gravity is

8 feet from the left end. Find the center of gravity of the rod and bodies combined.

Moments are taken about an axis through the left end perpendicular to the length of the rod. The solution may be arranged in convenient tabular form as follows.

	Mass	Moment arm	Moment
	16	0	0
	20	7	140
	4	8	32
	8	10	80
	12	14	168
Total	60 pounds.		420 foot-pounds

When the moment of 420 foot-pounds is divided by the total force of 60 pounds the result is 7 feet, which is the distance of the center of gravity from the left end of the rod.

Problems

1. Check the example above by moments about an axis which is 3 feet from the left end of the rod.
2. A straight horizontal rod is 12 feet long, weighs 40 pounds, and has its center of gravity at the middle. It carries 48 pounds on the left end, 72 pounds 4 feet from the left end, 60 pounds 7 feet from the left end, 30 pounds 1 foot from the right end. Find the center of gravity by moments about an axis through the left end and check by moments about an axis through the right end.

If the masses of the bodies in these problems are m_1, m_2, m_3, m_4, and so forth, and if the moment arms are x_1, x_2, x_3, and x_4, respectively, the sum of the moments is

$$M = m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4, \text{ and so forth,}$$

and the location of the center of gravity is given by

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}. \tag{1}$$

in which $\bar{x}$ is the distance of the center of gravity from the origin of moments.

195. Center of Gravity of Bodies in a Plane.—When it is desired to find the center of gravity of a number of bodies which have their centers of gravity in the same plane, moments must be computed about two axes. Figure 220 represents four bodies in the plane of the paper. The masses of these bodies are m_1, m_2, m_3, and m_4, respectively. If a point in the line AB of Fig.

220, I, is taken as the origin of moments, and if the forces of gravity on the masses have the direction of the arrows, the moment arms are x_1, x_2, x_3, and x_4, and the equation of the preceding article gives $\bar{x}$. To find the other coördinate of the center of gravity, the system may be rotated 90 degrees to the position of Fig. 220,

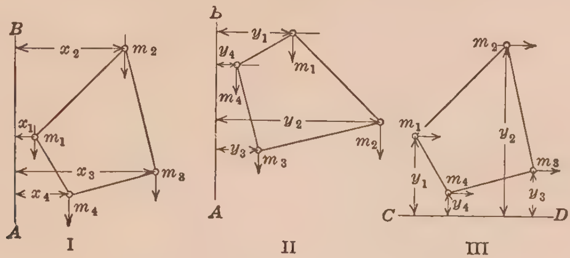


FIG. 220.—Center of gravity by moments.

II. It is better, however, to consider the system fixed in the position of Fig. 220, I, and to assume the forces to be rotated through an angle of 90 degrees to the direction shown in Fig. 220, III. If any point in the line CD is then taken as the origin of moments, the moment arms are y_1, y_2, y_3, and y_4, and

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4}.$$

For each of these calculations the axis of moments has been taken perpendicular to the plane of the paper. Instead of this, each axis may be regarded as parallel to the plane of the paper and the forces may be assumed perpendicular to that plane. In Fig. 220, I, the line AB may be taken as the axis of moments for finding $\bar{x}$. In fact, to find $\bar{x}$, any axis may be used, provided it lies in a plane which is normal to x_1, x_2, etc.

In finding the center of gravity, it is customary to speak of moments with respect to a plane. To find one coördinate of a center of gravity, multiply each mass by the distance of its center of gravity from some plane of reference. The sum of these moments divided by the sum of the masses gives the required distance.

Problems

1. A body is composed of three particles in the same plane

Mass	x	y
3	5	7
2	4	8
5	3	6

Find $\bar{x}$ and $\bar{y}$.

$$\bar{x} = \frac{3 \times 5 + 2 \times 4 + 5 \times 3}{3 + 2 + 5} = \frac{38}{10} = 3.8,$$

$$\begin{aligned} 3 \times 7 &= 21 \\ 2 \times 8 &= 16 \\ 5 \times 6 &= 30 \\ \hline 10\bar{y} &= 67 \\ \bar{y} &= 6.7 \end{aligned}$$

The second form of solution is preferable, especially when the numbers are large. It is still better to arrange the data and results in a single table, as shown in Problem 2, with the multiplication and equality sign omitted.

2. Find the coördinates of the center of gravity of

Mass	x	y	mx	my
12	4	3		
10	−3	4		
8	3	7		
	$\bar{x}$	=	1.4	
	$\bar{y}$	=	?	

3. Solve Problem 2 taking moments with respect to the planes $x = 2$, and $y = 3$.

196. Center of Gravity of a Plane Area.—The location of the point which is called the center of gravity of a plane area is important in the study of Strength of Materials. A plane area may be regarded as a plate of uniform thickness and density. In calculating the moment, it is customary to consider this plate to have unit mass per unit area.

The center of gravity of a few plane areas may be determined geometrically. If the area has a line of symmetry, the center of gravity lies in this line. If it has two lines of symmetry the center of gravity is at their intersection. The center of gravity of a circle is at the center. The center of gravity of the rectangle of Fig. 221 lies in the line of symmetry AB, which is midway between the ends. It also lies in the line of symmetry CD, which is midway between the sides. The center of gravity is, therefore, at the intersection of these lines.

A triangle, Fig. 222, may be regarded as made up of an infinite number of infinitesimal rectangles, such as AB, all of which are parallel to the base BH. Since the center of gravity of each of these rectangles is at the middle of its length, the center of gravity of the triangle must lie in the median CD. In a similar

manner, it may be shown that the center of gravity of the triangle lies in the median EF and in the median GH. It is proved by geometry that the three medians of a triangle intersect at a point which is two-thirds the length of each median from the

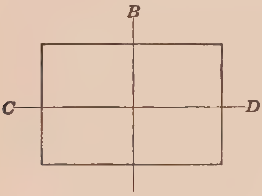


FIG. 221.—Center of gravity of rectangle.

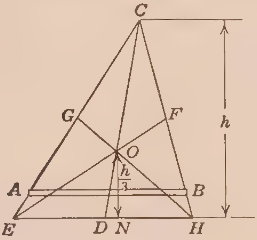


FIG. 222.—Center of gravity of triangle.

vertex of the triangle through which it passes. The line OD is one-third of CD. If H is the altitude of the triangle, measured perpendicular from the base, the perpendicular distance ON, from the center of gravity to the base is $\frac{h}{3}$.

The center of gravity of a parallelogram lies at the intersection of two lines parallel to the sides and midway between them.

Many important figures are made up of combinations of triangles, parallelograms, and circles.

Example

Find the distance of the center of gravity of a 6-inch by 5-inch by 1-inch angle section from the back of the legs.

The section may be divided into a 4-inch by 1-inch rectangle, and a 6-inch by 1-inch rectangle (Fig. 223). To find $\bar{x}$, moments may be taken ith respect to the axis Y-Y at the back of the 5-inch leg.

$$
\begin{array}{rcrcr}
4 & \times & 0.5 & = & 2 \\
6 & \times & 3.0 & = & 18 \\
\hline
10\bar{x} & & & = & 20 \\
\bar{x} & & & = & 2 \text{ inches}
\end{array}
$$

Problems

1. In the example above find $\bar{y}$. *Ans.* $\bar{y} = 1.5$ inches.
2. Solve Problem 1 by dividing the section into two 5-inch by 1-inch rectangles.
3. The lower base of a trapezoid (Fig. 224) is 10 inches, the upper base is 6 inches, and the altitude is 12 inches. Find the distance of the center of

gravity from the lower base by dividing the trapezoid into a rectangle and a triangle.

4. Solve Problem 3 by dividing the trapezoid into two triangles.

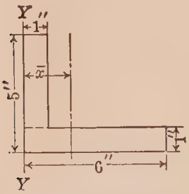


Fig. 223.—Center of gravity of angle section.

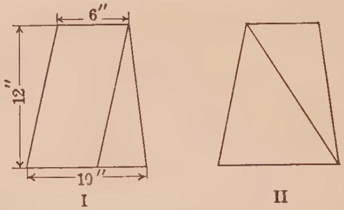


Fig. 224.—Center of gravity of trapezoid.

It is sometimes convenient to consider a given area as the difference between two areas. The angle section of Fig. 223 may be regarded as a 6-inch by 5-inch rectangle from which a 5-inch by 4-inch rectangle has been subtracted.

Area	Arm	Moment
30	3	90
− 20	3.5	− 70
$10\bar{x}$	=	20
$\bar{x}$	=	2

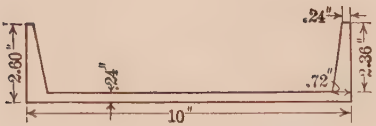


Fig. 225.—A channel section.

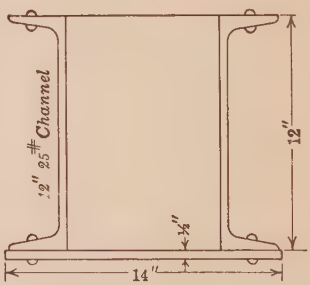


Fig. 226.—Plate and channel section.

Problems

5. Solve Problem 3 by regarding the trapezoid as the difference between a 10-inch by 12-inch parallelogram and a triangle with a 4-inch base.

6. A rectangular board 20 inches long and 18 inches wide has a circular hole, 8 inches in diameter, cut out. The center of the hole is 5 inches from the left end and 6 inches from the lower edge. Find the center of gravity of the remainder. Check by means of axes through the center of the hole.

$\bar{x}$ = 10.81 inches from the left end.

7. A 12-inch circular board has an 8-inch hole with its center 1 inch from the center of the board. Find the distance of the center of gravity from

the center of the board. Do not multiply by π to find the actual area of either circle.

8. Figure 225 represents a standard 10-inch, 15-pound channel section. Find the distance of the center of gravity of the section from the back of the web, and compare with handbook.
9. Look up the dimensions of a standard 15-inch, 33-pound channel, and calculate the area and the distance of the center of gravity from the back of the web. (For dimensions, see Cambria Steel, 1919 Edition, page 12 or Carnegie Pocket Companion, page 26. The curves and fillets are neglected and the section is assumed to be made up of rectangles and trapezoids as shown in Fig. 225 or by the figure on page 29 of Cambria.)
10. A section is made of two 12-inch, 25-pound channels and one 14-inch by ½-inch plate. Look up area of channels in handbook and compute the distance of the center of gravity of the section from the common surface of the plate and channels.

197. Center of Gravity by Integration.—In the preceding article, the center of gravity has been found by dividing the area into a finite number of parts, such as rectangles, triangles, or

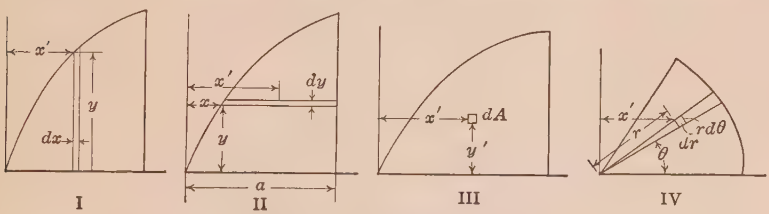


FIG. 227.—Elements of area.

circles, for each of which the area and location of the center of gravity is known. When the boundaries of the area are such that it can not be divided into these simple geometrical figures, the moment and area are found by integration. Figure 227 represents several areas. If x' is the distance from the plane with respect to which moments are taken to the *center of gravity* of an element of area dA, the moment of that element is $x'dA$ and the moment of the entire area is $\int x'dA$.

The total area is $\int dA$, therefore,

$$\bar{x} = \frac{\int x'dA}{\int dA} = \frac{\int x'dA}{A}. \tag{1}$$

$$\bar{y} = \frac{\int y'dA}{A}. \tag{2}$$

In Fig. 227, I, x' is the same as x of the curve at the intersection with the element of area. On the other hand, y' is one-half the y of the curve.

In Fig. 227, II, the element of area is horizontal and y' is equal to y. The length of this horizontal element is $a - x$, and $x' = x + \frac{a-x}{2} = \frac{a+x}{2}$. In Fig. 227, III, which is intended for double integration, $x' = x$ and $y' = y$.

Figure 227, IV, gives the polar coördinates. The element of area for double integration is $r\,d\theta\,dr$; $x' = r\cos\theta$; and $y' = r \sin\theta$.

Example

Find the center by gravity of a triangle of base b and altitude h by integration.

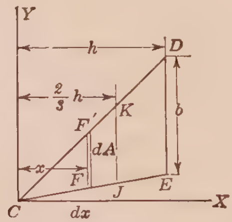


FIG. 228.—Center of gravity of triangle.

Figure 228 shows the triangle with the base parallel to the Y axis. The vertex C is placed at the origin of coördinates. The element of area is the strip FF', which is dx wide. From similar triangles, the length of this strip is found to be $\frac{bx}{h}$.

$$dA = \frac{bx}{h}dx; \tag{1}$$

$$\bar{x} = \frac{\frac{b}{h}\int x^2dx}{\frac{b}{h}\int xdx} = \frac{\frac{b}{h}\left[\frac{x^3}{3}\right]_0^h}{\frac{b}{h}\left[\frac{x^2}{2}\right]_0^h} = \frac{\frac{bh^2}{3}}{\frac{bh}{2}}, \tag{2}$$

$$\bar{x} = \frac{2}{3}h. \tag{3}$$

The denominator of the last term of Equation (2) is the known area of the triangle, which shows that the increment of area and the limits of integration were correctly taken.

The constant terms b and h occur in both numerator and denominator of Equation (2). The center of gravity would be correct if these were canceled before the first integration. It is best, however, to cancel no terms until

after the limits have been substituted in the final expressions. The two expressions then give the true moment and area and make it possible to compare these with any known values. *Variable quantities must never be canceled before integration.* If one limit is 0, a variable may be canceled after integration before the limits are substituted. If neither limit is 0, the division must not be made until after the limits have been substituted in the integral.

Problems

1. The lower base of a trapezoid is b, the upper base is c, and the height is h. By integration, find the distance of the center of gravity from the base and check by means of a triangle and a parallelogram.

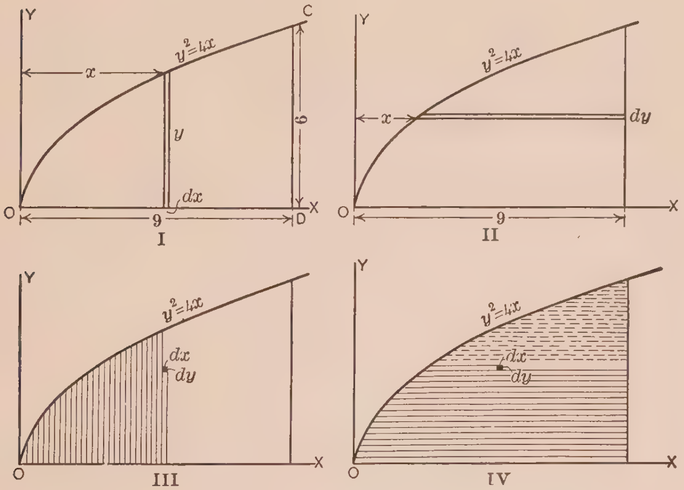


FIG. 229.—Center of gravity of area bounded by a parabola.

2. Find the distance from the Y axis of the center of gravity of the plane area bounded by the X axis, the parabola $y^2 = 4x$, and the ordinate $x = 9$ (Fig. 229, I).

The element of area is ydx, and the moment arm is x of the curve

$$\bar{x} = \frac{\int xydx}{\int ydx} \tag{4}$$

Since there are two variables in the integrals of Equation (4), one of these must be eliminated by substituting its value in terms of the other from the equation of the curve. First eliminate y by substituting $2\sqrt{x}$, and integrate between the limits for x. Then eliminate x and dx by substituting the values in terms of y and dy, and integrate between the limits for y. Compare the results.

3. Solve Problem 2 for $\bar{y}$, with the horizontal element of Fig. 229, II. Eliminate x and use y as the independent variable. The result should be greater than 2 and less than 3. Why?
4. Solve Problem 3 for $\bar{y}$, with the vertical element of Fig. 229, I.
5. Solve Problem 2 for $\bar{x}$ with the horizontal element of Fig. 229, II.
6. Solve Problem 2 for $\bar{x}$ by double integration, integrating first with respect to y. After substituting the limits in terms of x, compare with the second step of Problem 2. (See Fig. 229, III.)
7. Solve Problem 3 for $\bar{y}$ by double integration. Integrate first with respect to x (Fig. 229, IV). Compare with one of the single integrations.

When an area is integrated by double integration, the result of the first integration after the limits are substituted is equivalent to one of the first steps of a single integration. When rectangular coördinates are used, a double integration is seldom required in the solution of a problem of areas. With polar coördinates, on the other hand, double integration is generally preferable.

Table XXIV

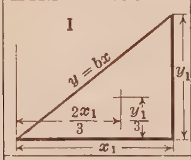
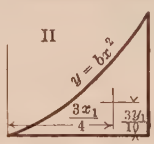
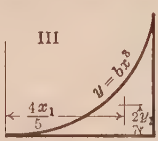
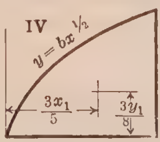
	Equation	Area	M_x	$\bar{x}$	$\bar{y}$
	$y=bx$	$\frac{x_1y_1}{2}$	$\frac{bx_1^3}{3}=\frac{x_1^2y_1}{3}$	$\frac{2x_1}{3}$	$\frac{y_1}{3}$
	$y=bx^2$	$\frac{x_1y_1}{3}$	$\frac{bx_1^4}{4}=\frac{x_1^2y_1}{4}$	$\frac{3x_1}{4}$	$\frac{3y_1}{10}$
	$y=bx^3$	$\frac{x_1y_1}{4}$	$\frac{bx_1^5}{5}=\frac{x_1^2y_1}{5}$	$\frac{4x_1}{5}$	$\frac{4y_1}{14}$
	$y=bx^{\frac{1}{2}}$	$\frac{2x_1y_1}{3}$	$\frac{2bx_1^{\frac{5}{2}}}{5}=\frac{2x_1^2y_1}{5}$	$\frac{3x_1}{5}$	$\frac{3y_1}{8}$

Fig. 230.—Center of gravity of some plane figures.

The student should always draw the diagram as in Fig. 229, construct the element of area, and determine how these ele-

ments should be joined together to form the entire figure. The algebra must always be interpreted in terms of the geometry and the geometry must be expressed in terms of the algebra.

Table XXIV gives the area, the location of the center of gravity, and the moment with respect to a plane through the left end, for some plane figures. The first two of these are of especial importance in finding the deflection of a beam by the method of Area Moments, and the others are sometimes used for this purpose.

Problems

8. Verify the values of Table XXIV.

9. Find the moment with respect to the Y axis of the area enclosed by the X axis, the line $y = bx$, and the ordinate $x = c$ and $x = d$. Solve by means of the Table XXIV and check by dividing the area into a triangle and a rectangle.

$$Ans.\ M_x = \frac{b(d^3 - c^3)}{3}.$$

10. Find $\bar{x}$ of the area bounded by the Y axis, the line $y = 6$, the hyperbola $xy = 12$, the line $x = 12$, and the X axis, using a vertical strip as the element of area.

$$Ans.\ \bar{x} = \frac{12 + \int xy\,dx}{12 + \int y\,dx} = \frac{12 + 12\int dx}{12 + \int \frac{12}{x}\,dx} = \frac{12 + 12\,[x]_2^{12}}{12 + 12[\log x]_2^{12}} = 3.94.$$

11. Find $\bar{x}$ for a 60-degree sector of a circle of radius a with the X axis as one of the bounding lines (Fig. 231). Solve by polar coördinates, integrating first with respect to r. The order of integration is immaterial in this problem, as the limits of one variable are independent of the other. When the limits are not independent in a problem of polar coördinates, integrate first with respect to r.)

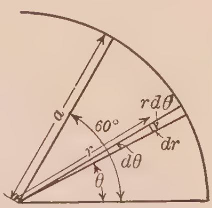


FIG. 231.—Center of gravity of a sector of a circle.

$$Ans.\ \bar{x} = \frac{\sqrt{3}a}{\pi} = 0.551a.$$

12. Using the value of $\bar{x}$ from Problem 11, find $\bar{y}$ without integrating.

$$Ans.\ \bar{y} = \frac{a}{\pi}.$$

13. Solve Problem 11 for $\bar{x}$ if the sector is so placed that the X axis bisects it. Compare with results of the preceding problems.

14. Find the center of gravity of segment of a circle of radius 10 bounded on one side by a straight line at a distance 5 from the center of the circle. Solve by rectangular coördinates, using strips parallel to the boundary line as increments of area. $Ans.\ \bar{x} = 7.05.$

Using only the half above the X axis and calling the radius a:

$$\bar{x} = \frac{\int xydx}{\int ydx} = \frac{\int (a^2 - x^2)^{\frac{1}{2}} xdx}{-a^2 \int \sin^2 \theta d\theta}$$

$$\bar{x} = \frac{\dfrac{-\left[\left(a^2 - x^2\right)^{\frac{3}{2}}\right]_{\frac{a}{2}}^{a}}{3}}{-a^2\left[\dfrac{\theta}{2} - \dfrac{\sin 2\theta}{4}\right]_{\frac{\pi}{3}}^{0}} = \frac{\dfrac{a^3\sqrt{3}}{8}}{a^2\left[\dfrac{\pi}{6} - \dfrac{\sqrt{3}}{8}\right]}.$$

The independent variable is changed in the denominator and might also be changed in the numerator. Why is the upper limit in the denominator 0 and not $\frac{\pi}{3}$? Explain the geometric meaning of each term in the denominator.

15. Solve Problem 14, by double integration with polar coördinates (Fig. 232, II).

16. Find the center of gravity of a semicircular area of radius a.

Ans. $\bar{x} = \frac{4a}{3\pi} = 0.4244a.$

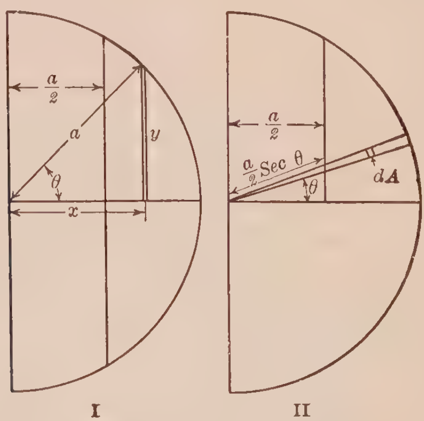


FIG. 232.—Center of gravity of a segment of a circle.

198. Center of Gravity by Experiment.—Sections of materials subjected to bending are frequently of such form that it is difficult to express the boundaries in mathematical symbols. For any such section, the center of gravity may be found by cutting it out of a sheet of card board or metal and balancing on a knife-edge. Where considerable accuracy is desired, it is best to cut out a short portion of the actual beam between two parallel

transverse planes and locate the center of gravity by moments on the beam of a delicate balance.

Figure 233, I, represents a body AG, which is balanced by the poise P. If the body is then turned end for end on the balance beam to bring the point A to the position A', while the opposite end remains stationary, the center of gravity is moved through the distance GG', which is twice its distance from the large end

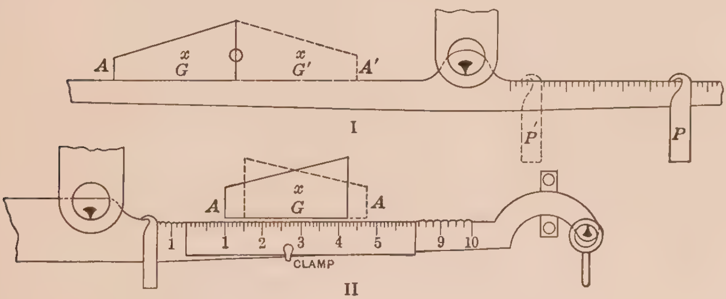


FIG. 233.—Center of gravity by balancing.

of the body. The poise P must be moved to the position P'. If the mass of the body is known in terms of the mass of the poise, the distance GG' may be calculated.

Problem

1. A plate of uniform thickness is placed on a light horizontal shelf which is attached to the beam of a balance. When the plate is turned end for end about a given line, a poise weighing 20 grams is moved 2.40 inches to balance. The plate weighs 32 grams. How far is the center of gravity from the line of reference? *Ans.* 0.75 inch.

Another method for finding the center of gravity is shown by Fig. 233, II.

The body is placed on the beam as before and moved till equilibrium is secured with some convenient weight on the opposite end. It is then turned end for end and moved along the scale beam until the same balance is secured, with all other weights unchanged. The center of gravity is now in the same position which it occupied before turning. If the position of any point such as A is noted before turning and again after turning, the difference of these two positions is twice the horizontal distance of A from the center of gravity. This may be done with great

accuracy on the beam of a platform scale by clamping to the beam a small steel scale for determination of the displacement of the points as shown in the figure.

Problem

2. A body is balanced on a scale beam. When turned around and again balanced, it is found that the point originally at the left end is displaced 3.32 inches. How far is the center of gravity from this end?

Ans. 1.66 inches.

CHAPTER XX

MOMENT OF INERTIA

199. Definition.—The moment of inertia of a body with respect to an axis is the sum of the products obtained by multiplying the mass of each particle of the body by the square of its distance from the axis. If m is the mass of any particle, and r is its distance from the axis,

$$I = \Sigma mr^2.$$

For a continuous body, the definition expressed mathematically is

$$I = \int r^2 dM, \qquad \text{Formula XXXIII}$$

in which I is the moment of inertia and dM is any element of mass.

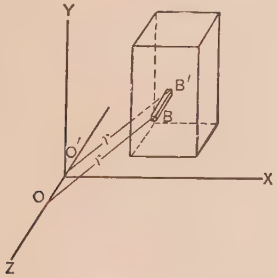


FIG. 234.—Element of volume.

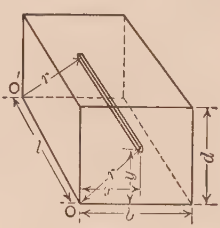


FIG. 235.—Moment of inertia of parallelopiped.

In Fig. 234, the moment of inertia is taken with respect to the Z axis. The element BB', which extends through the body parallel to the Z axis, is the element of mass of Formula XXXIII. This element of mass might have the form of a hollow cylinder of radius r and thickness dr, or it might be infinitesimal in three directions. The mass dM is the volume of this element dV multiplied by the density

$$dM = \rho dV.$$

If the moment of inertia is taken with respect to the Z axis and if z' is the length of an element parallel to the axis and extending entirely through the body, the expressions for dV and r are:

With rectangular coördinates,

$$dV = dx\,dy\,dz \text{ for triple integration;}$$
$$dV = z'dx\,dy \text{ for double integration;}$$
$$r^2 = x^2 + y^2.$$

With cylindrical or mixed coördinates with the Z axis as the axis of the cylinder

$$dV = r\,d\theta\,dr\,dz \text{ for triple integration;}$$
$$dV = z'r\,d\theta\,dr \text{ for double integration;}$$
$$dV = 2\pi\,rz'\,dr \text{ for single integration;}$$
$$r^2 = r^2.$$

With spherical coördinates with the Z axis as the axis of the sphere from which θ is measured,

$$dV = r^2 \sin\theta\,d\theta\,d\phi\,dr \text{ for triple integration}$$
$$dV = 2\pi r^2 \sin\theta\,d\theta\,dr \text{ for double integration}$$
$$r^2 = r^2 \sin^2\theta.$$

Problems

1. By double integration, find the moment of inertia of a rectangular parallelopiped of length l, width b, and height d with respect to an edge parallel to the length (Fig. 235).
Ans. $I = \frac{bdl}{3}(b^2 + d^2) = \frac{M}{3}(b^2 + d^2)$, in which M is the total mass.

2. Find the moment of inertia of a homogeneous solid cylinder of length l and radius a with respect to the axis of revolution. Solve by double integration with cylindrical coördinates. *Ans.* $I = \frac{\pi\rho l a^4}{2} = \frac{Ma^2}{2}$.

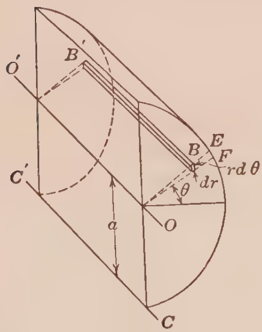


FIG. 236.—Moment of inertia of cylinder.

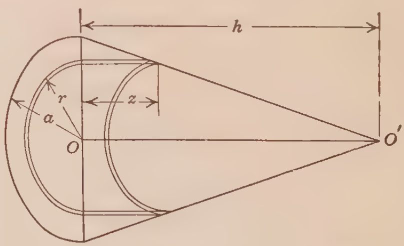


FIG. 237.—Moment of inertia of a cone.

The element of volume has a constant length l and cross-section $r\,d\theta\,dr$. The first integration with respect to r gives a wedge-shaped element between the planes whose traces on the front of Fig. 236 are the lines OE and OF. The second integration builds up the cylinder of a series of such wedges.

If θ is integrated first between 0 and 2π, the result is a hollow cylinder of radius r and thickness dr. The volume of this hollow cylinder is $2\pi rl\,dr$, and its moment of inertia with respect to the axis OO' is $2\pi\rho lr^3\,dr$, which might have been obtained directly without integrating.

3. Find the moment of inertia of a right cone of height h and radius of base a with respect to the axis of revolution, by a single integration using a hollow cylinder as the element of volume.

$$Ans.\ I = 2\pi\rho \int r^3 z\,dr = 2\pi\rho h \int_0^a \left(r^3 - \frac{r^4}{a}\right) dr = \frac{\pi\rho a^4 h}{10} = \frac{3Ma^2}{10}.$$

4. Find the moment of inertia of a homogeneous solid sphere of radius a with respect to a diameter. Use as the element of volume a ring of radius r and cross-section $dr\,dz$. Integrate first with respect to z. What is the form of the element after this integration?

$$Ans.\ I = \frac{2Ma^2}{5}.$$

5. Integrate Problem 4 first with respect to r. Show that the result of the first integration gives the moment of inertia of a disk of thickness dz. How would you solve the problem with a single integration and the results of Problem 2?

6. Solve Problem 3 by a single integration building the cone of flat disks parallel to the base and applying the results of Problem 2 for finding the moment of inertia of each disk.

The moment of inertia of each disk is $\frac{Ma^2}{2}$, in which $a = r$ and $M = \pi\rho r^2 dx$.

200. Radius of Gyration.—The radius of gyration may be defined algebraically by the equations:

$$Mk^2 = I, \tag{1}$$

$$k^2 = \frac{I}{M}, \tag{2}$$

in which k is the radius of gyration.

The radius of gyration is the distance from the axis at which the entire mass could be concentrated and leave the moment of inertia unchanged. In the case of a homogeneous solid cylinder with respect to the axis of revolution,

$$I = \frac{Ma^2}{2};$$

$$k^2 = \frac{a^2}{2};$$

$$k = \frac{a}{\sqrt{2}} = 0.7071a.$$

If the entire mass of a solid cylinder of radius a were condensed into a hollow cylinder of radius $0.707a$ and negligible thickness,

or into a single filament at a distance of $0.707a$ from the axis, the moment of inertia in each case would be the same as that of the solid cylinder.

Problems

1. A homogeneous solid cylinder is 24 inches in diameter. Find the radius of gyration with respect to the axis of the cylinder.

Ans. $k = 6\sqrt{2} = 8.48$ inches.

2. A rectangular parallelopiped is 6 inches by 8 inches by 12 inches. Find the radius of gyration with respect to one of the 8-inch edges.

Ans. $k = 7.75$ inches.

3. A solid cylinder which is 20 inches in diameter and weighs 40 pounds is coaxial with a cylinder which is 12 inches in diameter and weighs 60 pounds. Find the moment of inertia and the radius of gyration of the two cylinders with respect to their common axis.

Ans. $k = 5.55$ inches.

4. Find the radius of gyration of a homogeneous hollow cylinder of outside radius a and inside radius b with respect to the axis of revolution.

Ans. $k = \sqrt{\dfrac{a^2 + b^2}{2}}$.

5. By integration, find the moment of inertia of a homogeneous solid cylinder with respect to an element of the curved surface as an axis (CC', Fig. 236.)

Ans. $I = \frac{3}{2}Ma^2$.

6. Solve Problem 5 with the origin of coördinates at the axis of the cylinder. The element of area is $r\,d\theta\,dr$, in which r is measured from the axis of the cylinder, and the element of volume is $l\,r\,d\theta\,dr$. In the expression r^2dm, r is the distance from the axis of inertia to the element of mass. If this distance is represented by r_1,

$$r_1^2 = a^2 + r^2 + 2ra\cos\theta,$$

in which θ is the external angle between the radius r and the plane through the axis of inertia and the axis of the cylinder.

7. Find the moment of inertia of a homogeneous solid cylinder with respect to an axis which is parallel to the axis of the cylinder and is at a distance d from it. Solve with the origin of coördinates at the axis of the cylinder (Fig. 238).

Ans. $I = M\left(d^2 + \dfrac{a^2}{2}\right)$.

201. Transfer of Axis.—The integration of Problem 5 of the preceding article is laborious. The method of Problem 6 is much easier. If an attempt were made to solve Problem 7 with the origin of coördinates on the axis of inertia (through 0, Fig. 238) the integration would be found extremely difficult. It is advisable, therefore, to choose as axes of coördinates those axes which make the integration the simplest. All the advantages of this method, however, are secured by means of the prop-

osition for the transfer of the axis. If I_0 is the moment of inertia of a body with respect to an axis through its center of gravity, and I is the moment of inertia with respect to any other axis which is parallel to this axis through the center of gravity and at a distance d from it, the proposition states that

$$I = I_0 + Md^2$$

Formula XXXIV.

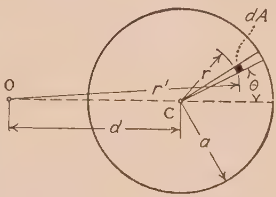


FIG. 238.

In Fig. 239, CC' is an axis through the center of gravity of a solid, and OO' is a parallel axis at a distance d from CC'. In Fig. 239, I, BB' is an element of volume parallel to CC' and OO'. Its coördinates with respect to CC' are (x,y). The coördinates of CC' with respect to OO' are (a,b).

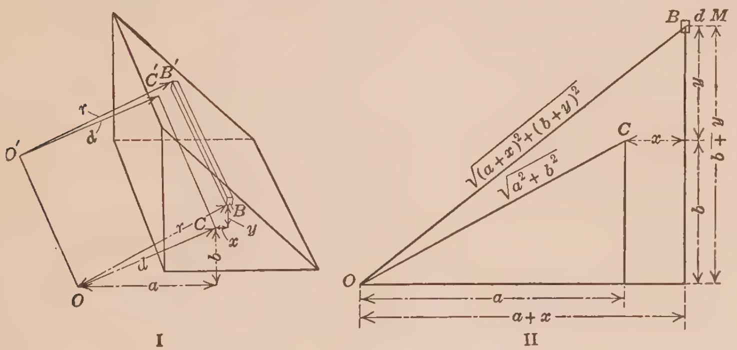


FIG. 239.—Transfer of axis.

$$d^2 = a^2 + b^2. \tag{1}$$

When the moment of inertia is taken with respect to CC', the value of r in the expression $\int r^2 dM$ is

$$r^2 = (a + x)^2 + (b + y)^2. \tag{2}$$

With respect to OO' the expression for the moment of inertia is

$$I = \int (a^2 + 2ax + x^2 + b^2 + 2by + y^2)dM. \tag{3}$$

$$I = \int (x^2 + y^2)dM + (a^2 + b^2)\int dM + 2a\int xdM + 2b\int ydM \tag{4}$$

The first term of the second member of Equation (4) is the moment of inertia with respect to CC'. The second term is $(a^2 + b^2)\,M$ which is Md^2.

Nothing so far in the derivation limits the line CC' to the center of gravity of the solid. Equation (4) applies to the transfer from any axis to any parallel axis, neither of which are through the center of gravity. The transfer is greatly simplified, however, if one of the axes passes through the center of gravity. In the second member of Equation (4), $\int xdM$ is the moment of the body with respect to a vertical plane through the axis CC' and $\int ydM$ is the moment with respect to the horizontal plane through this axis. If the vertical plane through CC' passes through the center of gravity of the body, the moment is zero and the third term of the second member of Equation (3) vanishes. If the horizontal plane through CC' passes through the center of gravity, the last term vanishes. If, therefore, the line CC' passes through the center of gravity of the body, the last two terms vanish and Equation (4) becomes

$$I = \int (x^2 + y^2)dM + (a^2 + b^2)\int dM = I_0 + Md^2.$$

Formula XXXIV.

Problems

1. Solve Problem 5 of the preceding article by means of transfer of axis.
2. Solve Problem 7 of the preceding article by means of transfer of axis.
3. A homogeneous solid sphere is 6 inches in diameter and weighs 40 pounds. Find its moment of inertia with respect to a diameter and find its moment of inertia with respect to an axis 20 inches from the center.

   Ans. $I_0 = 144$; $I = 16{,}144$.
4. A homogeneous solid sphere is 5 inches in diameter. Find the radius of gyration with respect to a diameter. Solve also for an axis 20 inches and for an axis 40 inches from the center.
5. If k_0 is the radius of gyration with respect to an axis through the center of gravity and k is the radius of gyration with respect to a parallel axis at a distance d from the center of gravity, prove that

$$k^2 = k^2{}_0 + d^2.$$

6. A rectangular parallelopiped is 10 inches long, 8 inches wide from left to right, and 6 inches high. The weight is 6 pounds. Find the moment of inertia with respect to an axis which is parallel to the 10-inch edges, is 5 inches below the plane of the lower surface and is 12 inches to the left of the left vertical plane. Solve by the complete Equation (4), with the lower left 10-inch edge as the axis CC'.

   Ans. $I = 200 + 169 \times 6 + 2 \times 12 \times 4 \times 6 + 2 \times 5 \times 3 \times 6 = 1{,}970$.
7. Solve Problem 6 by transfer from the axis through the center of gravity.

   Ans. $I = 50 + 1{,}920 = 1{,}970$.

8. Find the moment of inertia of a rectangular parallelopiped with respect to an axis through the center of gravity, by a transfer of the answer of Problem 1 of Article 199 to the center of gravity.

202. Moment of Inertia of a Plane Area.—The moment of inertia of a plane area may be defined mathematically by the expression

$$I = \int r^2 dA. \tag{1}$$

It is equivalent to the moment of inertia of a thin plate of mass unity per unit area and of such small thickness that the square of the thickness is negligible compared with the square of the other dimensions.

There are two important cases of the moment of inertia of a plane area; in the first case the axis lies in the plane of the area; in the second case the axis is normal to the plane.

The moment of inertia of a plane area with respect to an axis in its plane is an important constant in all problems concerning the strength or deflection of beams or columns. This moment of inertia is represented by the letter I.

The moment of inertia of a plane area with respect to an axis perpendicular to its plane is called the *polar moment of* inertia. The polar moment of inertia is an important factor in all problems concerning the strength of shafting in torsion and the amount of twist of such shafts. It is represented by the letter J.

The polar moment of inertia of a plane area is equivalent to the moment of inertia of a solid plate of the same dimensions and of such thickness that the product of the thickness and density is unity. The radius of gyration of a plane area is given by:

$$k^2 = \frac{I}{A}, \tag{2}$$

$$k^2 = \frac{J}{A}, \tag{3}$$

which are the same as in the case of solids with the area used instead of the mass.

Formula XXXIV for the transfer of axes is modified in the same way,

$$I = I_0 + Ad^2. \tag{4}$$

Problems

1. By integration find the moment of inertia of a rectangle of breadth b and depth d with respect to the side b.

Ans. $I = \dfrac{bd^3}{3}.$

2. By transfer of axis find the moment of inertia of a rectangle of sides b and d with respect to an axis in the plane of the area parallel to b and passing through the center of the rectangle.

$$Ans. \ I = \frac{bd^3}{12}.$$

3. Find the moment of inertia and radius of gyration of a 6-inch by 10-inch rectangle with respect to an axis 2 inches outside the rectangle and parallel to a 6-inch edge. Solve by means of the answer to Problem 2 and transfer of axis. Check by the answer to Problem 1, subtracting the moment of inertia of a 6-inch by 2-inch rectangle from that of a 6-inch by 12-inch rectangle. *Ans.* $I = 3{,}440$ inches.^{4}

4. By integration with polar coördinates find the expression for the moment of inertia of a circular area of radius a with respect to a diameter.

$$Ans. \ I = \frac{\pi a^4}{4}.$$

The answers to Problems 1, 2, and 4 should be memorized.

5. Find the radius of gyration of a circular area of radius a with respect to a diameter.

$$Ans. \ \frac{a}{2}.$$

6. Find the polar moment of inertia of a circle of radius a with respect to an axis through its center. Show that the radius of gyration is the same as that of a solid cylinder of the same radius. Find the relation between the polar moment of inertia and the moment of inertia of the same circle with respect to a diameter. *Ans.* $J = 2I$.

7. Find the moment of inertia of a triangle of base b and altitude h with respect to an axis through the vertex parallel to the base. Solve by integration.

$$Ans. \ I = \frac{bh^3}{4}.$$

8. By transfer of axis find the moment of inertia of the triangle of Problem 7 with respect to an axis through the center of gravity parallel to the base.

$$Ans. \ I_0 = \frac{bh^3}{36}.$$

9. By transfer of axis find the moment of inertia of a triangle of base b and altitude h with respect to the base.

$$Ans. \ I = \frac{bh^3}{12}.$$

10. Find the moment of inertia of the trapezoid of lower base 16 inches, upper base 10 inches, and height 6 inches with respect to the lower base. Solve by dividing the trapezoid into a parallelogram and a triangle and use the answers to Problems 1 and 9.

11. Find the moment of inertia of the trapezoid of Problem 10 with respect to an axis through the center of gravity parallel to the base.

12. Find the moment of inertia of a 6-inch by 4-inch by 1-inch angle section with respect to an axis through the center of gravity parallel to the 4-inch leg.

Divide the section into two rectangles. Find the moment of inertia of each rectangle with respect to a line through *its* center of gravity parallel to

the 4-inch leg and transfer to the required axis. Dividing the section into a 6-inch by 1-inch and a 1-inch by 3-inch rectangle, Fig. 240,

$$\frac{1 \times 6^3}{12} + 6\left(\frac{5}{6}\right)^2 = 22\frac{1}{6},$$
$$\frac{3 \times 1^3}{12} + 3\left(\frac{10}{6}\right)^2 = 8\frac{7}{12},$$
$$I_0 \text{ for axis } 2\text{–}2 = 30\frac{3}{4} = 30.75 \text{ inches.}^4$$

13. Solve Problem 12 by another method. Find first the moment of inertia with respect to the right edge of the 4-inch leg and then transfer *to* the center of gravity.

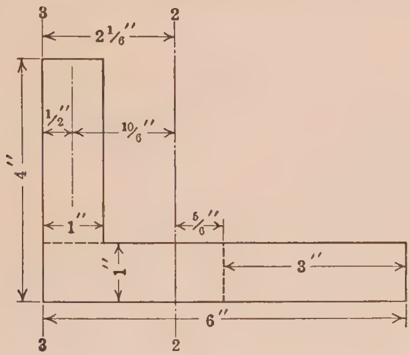


FIG. 240.

14. Solve Problem 12 for the moment of inertia with respect to the axis through the center of gravity parallel to the 6-inch leg. Compare the results with the handbook.

15. Find the moment of inertia and radius of gyration of a 5-inch by 3½ inch by ½-inch angle section for an axis through the center of gravity parallel to the 5-inch leg. To avoid fractions, solve for a section with twice these dimensions and then reduce. Compare with the handbook.

16. A plate-and-angle section (see handbook) is made of one 12-inch by ½-inch plate and four 6-inch by 4-inch by ½-inch angles with the 6-inch legs perpendicular to the plate and with the backs of the 6-inch legs ⅛ inch beyond the top and bottom of the plate. Look up the location of the center of gravity and the moment of inertia of the angles in the handbook and calculate the moment of inertia and the radius of gyration with respect to two lines of symmetry. Compare with Carnegie Pocket Companion.

17. Look up in the handbook the formula for the moment of inertia of a standard channel for an axis through the center of gravity perpendicular to the web. Derive this formula by means of the formulas for rectangles and triangles and Formula XXXIV. The slope of the standard channels is one-sixth.

18. Calculate the moment of inertia of a 10-inch, 15-pound standard channel section and compare the result with the handbook.

19. Find the moment of inertia of the plate and channel section of Fig. 226 with respect to the axis through the center of gravity parallel to the plate and also with respect to the axis through the center of gravity perpendicular to the plate.

203. Product of Inertia.—The expression $\int xydA$ is called the *product of inertia* of the area. It is represented algebraically by the letter H.

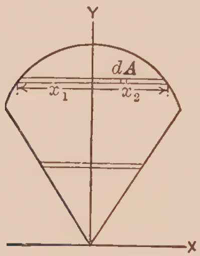


FIG. 241.—Symmetrical section.

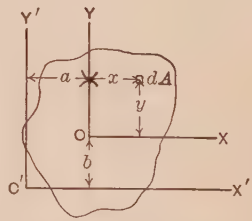


FIG. 242.—Transfer of axes for product of inertia.

If an area is symmetrical with respect to either one of a pair of rectangular axes, its product of inertia with respect to that pair of axes is zero. Figure 241 represents an area which is symmetrical with respect to the y axis. If this is integrated first with respect to x,

$$H = \frac{1}{2}\int \Big[x^2\Big]_{x_1}^{x_2} y\,dy = \frac{1}{2}\int [x_2^2 - x_1^2]y\,dy.$$

If the area is symmetrical with respect to the Y axis, the lower limit x_1 is numerically equal and opposite in sign to the upper limit x_2, and the squares are the same in magnitude and sign; consequently the term in the brackets vanishes and

$$H = 0.$$

When the product of inertia is known with respect to a pair of rectangular axes through the center of gravity of an area, it may be calculated for a second pair of parallel axes in the plane of the area by a formula similar to XXXIV for the transfer of moments of inertia.

Let OX, OY, Fig. 242, be the original pair of axes through the center of gravity, and let (x, y) be the coördinates of an element dA with reference to these axes. Let $O'X'$, $O'Y'$ be a new pair of parallel axes. Let (a, b) be the coördinates of the center of gravity of the area with respect to the new axes.

If H is the product of inertia with respect to the new axes,

$$H = \int (a + x)(b + y)dA. \tag{1}$$

$$H = ab\int dA + b\int xdA + a\int ydA + \int xydA. \tag{2}$$

$$H = abA + 0 + 0 + H_0. \tag{3}$$

where H_0 is the product of inertia with respect to the axes through the center of gravity. Equation (3) is easily remembered from Formula XXXIV, replacing the square by the product.

If the center of gravity falls in the first or third quadrant with respect to the axes for which the product of inertia is desired, the product abA is positive, and H will be positive unless H_0 is negative and numerically greater than abA. If the center of gravity falls in the second quadrant abA is negative since a is negative; if it falls in the fourth quadrant abA is negative because b is negative.

Problems

1. By integration find the product of inertia of a rectangle of base b and altitude d with respect to the lower and left edges as axes. Check by Equation (3).

Ans. $H = \frac{b^2d^2}{4}$.

2. Solve Problem 1 for the lower and right edges as axes.

Ans. $H = -\frac{b^2d^2}{4}$.

3. By integration find the product of inertia of a 6-inch by 8-inch right-angled triangle with respect to the edges as axes (Fig. 243).

Ans. $H = 96$ inches.[4]

4. By transfer of axes find the product of inertia of the triangle of Problem 3 with respect to the axes 1–1, 2–2 through the center of gravity.

Ans. $H_0 = 96 - 128 = -32$ inches.[4]

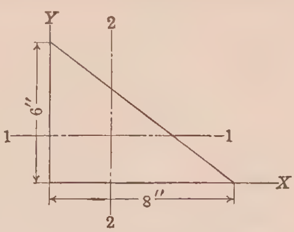


FIG. 243.—Product of inertia of a right triangle.

5. Find the product of inertia of a 6-inch by 5-inch by 1-inch angle section with respect to axes through the center of gravity parallel to the legs. The section may be divided into two rectangles. The product of inertia of each of these with respect to the axes through their center of gravity is zero. Transferring to the axes 1–1, 2–2 and adding

$$H_0 = 4 \times (-1.5) \times 1.5 + 0 + 6 \times 1 \times (-1) + 0 = -9 - 6 = -15 \text{ inches.}^4$$

The problem may be solved more readily in another way. Since 3–3 is an axis of symmetry for the horizontal leg, the product of inertia of this rec-

tangle for 3–3 and any other axis (as 4–4) is zero. Since axis 4–4 is a line of symmetry for the vertical leg, the product of inertia of this rectangle for axes 4–4 and 3–3 is zero. The product of inertia for the entire section for axes 3–3 and 4–4 is the sum of products of inertia of the separate rectangles, therefore, $H = 0$. When the product of inertia is transferred from 3–3, 4–4 to axes 1–1, 2–2 the equation is

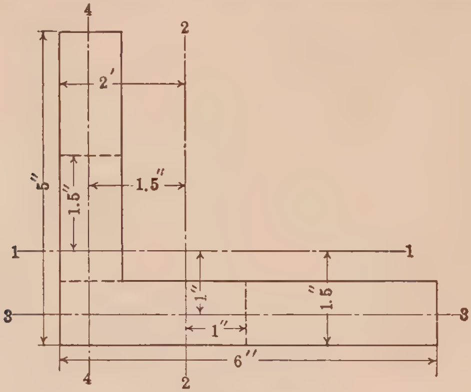


FIG. 244.

$$0 = H_0 + 10 \times 1.5 \times 1; \; H_0 = -15 \text{ inches}^4$$

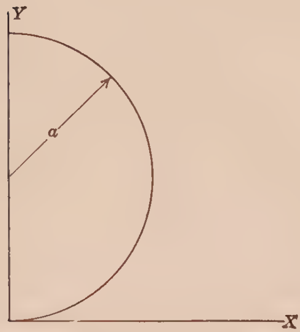


FIG. 245.—Product of inertia of semicircle.

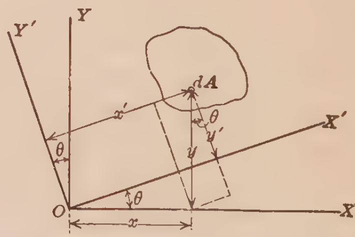


FIG. 246.—Change of direction of axis.

6. Find the product of inertia of an 8-inch by 6-inch by 1-inch angle section with respect to axes through the center of gravity parallel to the legs.
7. A semicircular area of radius a is in the position shown in Fig. 245. Find the product of inertia with respect to the X and Y axes.

Ans. $H = \dfrac{2a^4}{3}$.

The product of inertia has no physical significance, but is a convenient mathematical tool in finding the moment of inertia of a plane area with respect to any axis, as will be seen in the article which follows.

204. Change of Direction of Axis for Moment of Inertia.— By means of Formula XXXIV, moment of inertia may be transferred from one axis to a parallel axis. It is frequently necessary to find the moment of inertia with respect to an axis which is inclined to the principal lines of the figure in such a way that the solution by direct integration is difficult. If the moment of inertia of an area is known for any two axes in the plane at right angles to each other, the moment of inertia for any other axis at a known angle with these axes may be calculated.

Figure 246 represents any area in the XY plane. The moment of inertia of this area with respect to the X axis may be designated by I_x and the moment of inertia with respect to the Y axis may be designated by I_y.

$$I_x = \int y^2 dA;$$
$$I_y = \int x^2 dA.$$

The line OX' is a new axis which makes an angle θ with the X axis and OY' is an axis at right angles to OX'. The coördinates of an element of area dA with respect to these new axes are $(x'y')$.

The moment of inertia of the area with respect to OX' is

$$I = \int y'^2 dA. \tag{1}$$

From the geometry of the figure

$$y' = y \cos \theta - x \sin \theta. \tag{2}$$
$$I = \int (y^2 \cos^2 \theta - 2xy \cos \theta \sin \theta + x^2 \sin^2 \theta) dA, \tag{3}$$
$$I = I_x \cos^2 \theta + I_y \sin^2 \theta - 2 \cos \theta \sin \theta \int xy\, dA, \tag{4}$$
$$I = I_x \cos^2 \theta + I_y \sin^2 \theta - H \sin 2\theta. \tag{5}$$
$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - H \sin 2\theta. \tag{6}$$

Problems

1. Find the moment of inertia of a 4-inch by 3-inch rectangle (Fig. 247) with respect to an axis through the lower left corner making an angle of 20 degrees with the 4-inch edge.

$$I_x = 36 \text{ inches,}^4\ I_y = 64 \text{ inches,}^4\ H = 36 \text{ inches.}^4$$
$$I = \frac{36 + 64}{2} + \frac{36 - 64}{2} \cos 40° - 36 \sin 40°.$$
$$I = 50 - 14 \times 0.7660 - 36 \times 0.6428 = 16.14 \text{ inches.}^4$$

2. Solve Problem 1 if the axis is 20° below direction of the 4-inch edge.

Ans. $I = 62.42$ inches.^{4}

3. Find the moment of inertia of a 4-inch by 3-inch rectangle with respect to a diagonal by means of Equation (5) and check by means of the moment of inertia of two triangles with respect to their common base.
4. Find the moment of inertia of a 4-inch by 4-inch by ½-inch angle section of Fig. 248 with respect to axis 3–3, through center of gravity of section. Take I_x from the table in the handbook. *Ans.* $I = 2.29$ inches.[4]

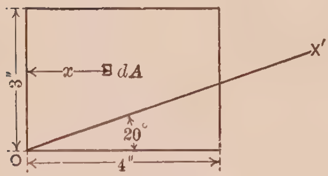


FIG. 247.—Moment of inertia with respect to OX'.

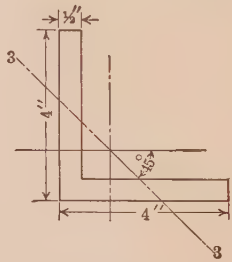


FIG. 248.—Axis of minimum moment of inertia.

205. Change of Direction of Axes for Product of Inertia.—

To derive the expression for the product of inertia for the axes OX', OY' of Fig. 246 the fundamental integral is

$$H' = \int x'y'dA. \tag{1}$$

$$H' = \int (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta)dA, \tag{2}$$

$$H' = (\cos^2 \theta - \sin^2 \theta) \int xy\, dA + \cos \theta \sin \theta \int (y^2 - x^2)dA, \tag{3}$$

$$H' = H \cos 2\theta + \frac{I_x - I_y}{2} \sin 2\theta. \tag{4}$$

When the expression to the right of the equality sign in Equation (4) is made equal to zero

$$\tan 2\theta = \frac{2H}{I_y - I_x} \tag{5}$$

Equation (5) gives the angle at which the product of inertia is zero.

Problems

1. In the 4-inch by 3-inch rectangle of Fig. 247 what will be the angle between OX' and the 4-inch edge if the product of inertia with respect to OX' and the axis through O normal to it is zero? *Ans.* $\theta = 34° 22'$.
2. Find the direction of the pair of axes through the center of gravity of the 6-inch by 5-inch by 1-inch angle section of Fig. 244 for which the product of inertia is zero.

206. Direction of Axis for Maximum Moment of Inertia.—From Equation (6) of Article 204 the moment of inertia with respect to an axis at an angle θ with the X axis is

$$I = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\theta - H \sin 2\theta. \qquad (1)$$

The direction of the axis for maximum or minimum moment of inertia is found by differentiating Equation (1) with respect to θ and equating the derivative to zero.

$$\frac{dI}{d\theta} = (I_y - I_x) \sin 2\theta - 2H \cos 2\theta, \qquad (2)$$

from which the condition of maximum or minimum is

$$\tan 2\theta = \frac{2H}{I_y - I_x}. \qquad (3)$$

A comparison of Equation (3) with Equation (5) of the preceding article shows that the condition for maximum and minimum moment of inertia is identical with the condition for zero product of inertia. There are two solutions for Equation (3), which give values of 2θ differing by 180 degrees and values of θ differing by 90 degrees. One of these gives the direction of the axis for which the moment of inertia is a maximum and the other gives the direction of the axis for which the moment of inertia is a minimum.

Since the product of inertia for an axis of symmerty is zero, the moment of inertia with respect to an axis of symmetry is greater or less than the moment of inertia for any other axis through any given point in its line.

The line which bisects the angle between the legs of an angle section of equal legs is a line of symmetry and the moment of inertia for this axis is greater than for any other axis through the center of gravity, while the moment of inertia for the axis at right angles to this line of symmetry (the axis 3–3 of Fig. 248) is smaller than that for any other axis through the center of gravity.

Problems

1. Find the maximum and minimum moment of inertia for axes through one corner of the rectangle of Problem 1 of Article 204.
2. Find the maximum and minimum moments of inertia for axes through the center of gravity of a 6-inch by 5-inch by 1-inch angle section. Compare the results with the values in the handbook for an angle section of one-half these dimensions.

The maximum and minimum moments of inertia of an area for axes through a given point are called the *principal moments of inertia*, and the corresponding axes are called the principal axes. If one of the principal moments of inertia is known, it is often easy to find the other by means of a simple relation. This is

$$I_{\max} + I_{\min} = I_x + I_y = J. \tag{4}$$

The sum of the moments of inertia of a plane area for any pair of rectangular axes in the plane is equal to the polar moment of inertia for their point of intersection.

Let one of these axes be used as the X axis.

$$I_x = \int y^2 dA. \tag{5}$$

If the other rectangular axis is used as the Y axis,

$$I_y = \int x^2 dA. \tag{6}$$

For the polar moment of inertia $r^2 = x^2 + y^2$,

$$J = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA. \tag{7}$$

Problems

3. Find the maximum moment of inertia of a 5-inch by 3-inch by ½-inch angle section for an axis through the center of gravity. Get I_x and I_y from the handbook, and calculate min. I from the area and the least radius of gyration. Solve for max. I by means of Equation (4).

4. Find the least and greatest moment of inertia and radius of gyration of a semicircular area of radius a with respect to axes in its plane passing through the end of the diameter which bounds it.

The moment of inertia of a square is the same for every axis through its center of gravity. In Equation (1), $I_x - I_y = 0$ and $H = 0$ for the diagonals and for axes parallel to the sides. Starting from either pair of these axes, there remains only the first term $I = \dfrac{I_x + I_y}{2} = I_x$.

It may also be proved that the moment of inertia is the same for every axis through the center of gravity of an equilateral triangle or any other regular polygon. The axes for maximum and minimum moment of inertia make angles of 90 degrees with each other. If θ is the angle which the first principal axis makes with the X axis, the next principal axis makes an angle of 90 degrees plus θ; the third principal axis makes an angle 180 degrees plus θ, and is, therefore, identical with the first. Like-

wise, the fourth axis is identical with the second. There are, therefore, only two principal axes through any given point in a plane figure. An axis of symmetry is a principal axis. An equilateral triangle has three axes of symmetry, a square has four, a regular pentagon has five, and so forth. Since it is possible to have only one maximum and one minimum, and the *mathematical* conditions for maxima and minima are fulfilled more than twice, it is evident that the moment of inertia is the same for all directions of the axis.

207. Moment of Inertia of a Prism or Pyramid.—The moment of inertia of any solid may be found by triple integration with an element which is infinitesimal in three directions, or by double integration with an element which is infinitesimal in two directions and extends entirely through the mass in the direction of the axis.

It is often better to use as the element of volume a thin plate or disk which is infinitesimal in one direction only, provided the moment of inertia of this element is known with respect to an axis which is parallel to the axis of inertia and passes through the center of gravity of the element.

Problems

1. Find the moment of inertia of a right pyramid of height h, with a square base of side b, with respect to an axis through the vertex perpendicular to the base.

The element of the volume is the square plate of thickness dx. Its volume is Adx, in which A is the area of the section. From similar solids (Fig. 249), in which

$$A = \frac{b^2x^2}{h^2}.$$

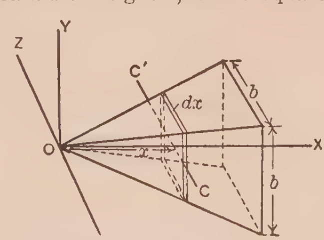


FIG. 249.—Moment of inertia of pyramid.

As each side is $\frac{bx}{h}$, its polar moment of inertia with respect to the X axis is $\frac{\rho b^4x^4}{6h^4}dx$. The total moment of inertia is the sum of that of the several plates.

$$I = \frac{\rho b^4x^5}{30h^4} = \frac{\rho b^4h}{30} = \frac{Mb^2}{10}.$$

2. Find the moment of inertia of a right pyramid, the base of which is a hexagon of side a, with respect to an axis through the vertex perpendicular to the base.

If it were desired to find the moment of inertia with respect to the Z axis OZ of Fig. 249, the moment of inertia of the plate could first be found with respect to the axis CC' and then transferred to OZ. The moment of inertia of the plate is the same as that of the area of the plate with respect to a line in its plane multiplied by the thickness dx and the density. The moment of inertia of the square plate of Fig. 249 with respect to CC' is

$$I_0 = \frac{\rho b^4 x^4}{12h^4}\,dx;$$

$$Md^2 = \frac{\rho b^2 x^4}{h^2}\,dx,$$

in which I_0 and Md^2 have the meaning of Formula XXXIV.

$$I = \rho\left(\frac{b^4x^5}{60h^4} + \frac{b^2x^5}{5h^2}\right) = \frac{\rho b^2 h}{5}\left(\frac{b^2}{12} + h^2\right).$$

Problems

3. Find the square of the radius of gyration of a right pyramid 24 inches high with base 12 inches square with reference to an axis through the vertex parallel to the base. *Ans.* $k^2 = 352.8$ inches.^{2}

4. Find the moment of inertia of a right cylinder of radius a and length l with respect to an axis perpendicular to the axis of the cylinder through the center of one end.

Ans. $k^2 = \dfrac{l^2}{3} + \dfrac{a^2}{4}.$

The answer of Problem 4 shows that the square of the radius of gyration consists of two terms. The first term is the square of the radius of gyration of a thin rod with respect to an axis which passes through one end and is perpendicular to the length of the rod, and the other term is the square of the radius of gyration with respect to a diameter. The moment of inertia of any solid of constant cross-section which ends with parallel planes perpendicular to its length (any right prism or cylinder) may be computed in the same way. Expressed algebraically

$$k^2 = k_l^2 + k_A^2, \tag{1}$$

in which k_l is the radius of gyration of the prism or cylinder regarded as a thin rod, and k_A is the radius of gyration of a cross-section regarded as a thin plate.

Figure 250 represents a triangular prism with its axis parallel to the X axis. It is desired to find the moment of inertia with

respect to the Z axis. The element of volume of length BB' and cross-section $dxdy$ extends entirely through the prism parallel to the Z axis. If dA is the area of the surface of length BB' and height dy the moment of inertia is

$$I = \rho \int \int (x^2 + y^2) dx dA, \tag{2}$$

$$I = \rho \int \int x^2 dx\, dA + \rho \int \int y^2 dx\, dA \tag{3}$$

when Equation (3) is integrated first with respect to dA, x remains unchanged. The integration consists of piling up

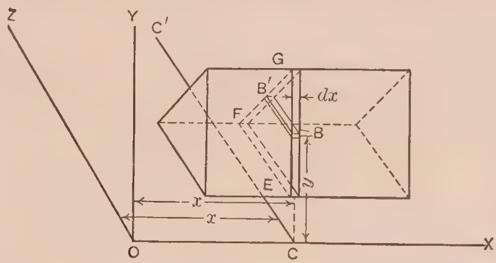


FIG. 250.—Moment of inertia of prism.

elements of the form of BB' from the bottom to the top of the section EFG between vertical planes at a distance dx apart. The result of the first integration is

$$I = \rho \int x^2 A\, dx + \rho \int I_A dx \tag{4}$$

in which A is the area of the section, and I_A is the moment of inertia of the plane area with respect to the axis CC' in the XZ plane parallel to the Z axis.

Equation (4) applies to a solid of any form whatever, and is not limited to a prism as shown in the figure. If the line OX passed through the center of gravity of all the sections the result would be an example of Formula XXXIV as in Problems 3 and 4.

If the solid is a prism or cylinder with the axis parallel to the X axis, A is constant and I_A is constant; then

$$I = \rho A \int x^2 dx + \rho I_A \int dx. \tag{5}$$

The first term of the last member of (5) is the moment of inertia of a thin rod with respect to an axis perpendicular to its length. The second member is equal to $\rho l A k_A^2 = M k_A^2$, which proves Equation (1).

Problems

5. Find the moment of inertia of a right cylinder 18 inches long and 12 inches in diameter with respect to an axis in the plane of one end and tangent to the cylinder. *Ans.* $I = 153M$.
6. Find the moment of inertia of a prism 6 inches square and 24 inches long with respect to an axis in the plane of one end perpendicular to the end of a diagonal. *Ans.* $I = M(192 + 21)$.

208. Moment of Inertia by Experiment.—The moment of inertia of a solid may be found from its effect upon the time of vibration of a torsion pendulum. The time of vibration of a torsion pendulum is given by

$$T^2 = KI, \tag{1}$$

in which T is the time of a single vibration or of a complete period (with K varying accordingly), and K is a constant which

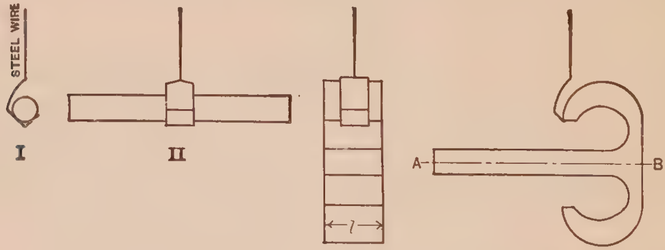


FIG. 251.—Moment of inertia by torsional vibration.

depends upon the length, diameter, and modulus of rigidity of the supporting wire. The factors which make up K need not be determined separately, as the entire term may be obtained by substitution from the time of vibration of a body of known moment of inertia. Figure 251, I and II, shows a uniform solid circular rod of bronze or brass in a horizontal position on a light support, which is suspended by a single steel wire. Figure 251, III and IV, shows a second body on the same support. It is desired to find the moment of inertia of this second body about a line through its center of gravity perpendicular to the line AB. If the body can be so supported that AB is horizontal, it will rotate about the desired axis since the center of gravity of the combined body and support must fall directly under the axis of the wire, and if the support is relatively small this combined center of gravity will practically coincide with that of

the body, even if the support does not hang in exactly the position which it occupies when it is not loaded.

When the moment of inertia of the support is small, as in Fig. 251, the unknown moment of inertia is calculated from

$$\frac{I_A}{I_C} = \frac{T_A^2}{T_C^2}, \tag{1}$$

in which the subscript A refers to the body and the subscript C refers to the cylinder.

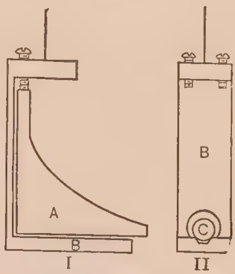


FIG. 252.—Support for torsional vibration.

It is not generally practicable to use a very light support and get the body into the desired position. Figure 252, I, shows the side elevation of a relatively heavy support, which carries the unknown body. Figure 252, II, shows the end elevation of the same support carrying the known cylinder. The time of vibration is taken with the support alone, with the support and the known body, and with the support and the unknown body.

$$T_B^2 = KI_B, \tag{2}$$
$$T_C^2 = K(I_C + I_B), \tag{3}$$
$$T_A^2 = K(I_A + I_B), \tag{4}$$

in which T_C is the time with the support and known cylinder, T_B is the time with the support alone, and T_A is the time with the support and the unknown body.

$$I_A = \frac{(T_A^2 - T_B^2)I_C}{T_C^2 - T_B^2}. \tag{5}$$

Problems

1. The time of vibration of a given torsion pendulum with the support alone is 0.46 second; with the support loaded with a cylinder 10 inches long and ½ inch in diameter it is 0.87 second; with an unknown body in place of the cylinder it is 0.94 second. The cylinder weighs 0.556 pound and the body 1.25 pounds. Find the moment of inertia and radius of gyration of the body. *Ans.* $k = 2.14$ inches.
2. Under what conditions may the unknown moment of inertia be accurately determined without getting the time of vibration of the support?
3. If any clamp screws are used in the support, they should be vertical. Why?

209. Moment of Inertia of a Plane Section.—The principles of the preceding article afford a convenient means of finding the moment of inertia of any section. A short portion of the material may be cut out between two parallel planes which are perpendicular to the length of the rod. The area of cross-section may be computed from the weight and specific gravity of the portion. The center of gravity may be determined by the method of moments as given in Article 198. The moment of inertia of the portion is found by the method of the preceding article. This moment of inertia divided by the mass gives k^2.

From Equation (1) of Article 207, the square of the radius of gyration of a prism is equal to the sum of the squares of the radius of gyration as a thin rod and the radius of gyration as a thin plate. If the length of the portion, for instance, is $\frac{1}{2}$ inch, the square of the radius of gyration as a thin rod is $\frac{1}{96}$, and the square of the radius of gyration of the section is the square of the radius of gyration of the prism minus $\frac{1}{96}$.

Problems

1. In the case of the unknown section of Fig. 251, III and IV, the length l is 1 inch, the weight in air 1.524 pounds, the weight in water 1.326 pounds. The water was at the temperature at which the density is 62.2 pounds per cubic foot. What is the area of cross-section? *Ans.* 5.50 square inches.

2. On a torsion pendulum with a light support the body in the position shown made 100 vibrations in 83.2 seconds. A rod $\frac{1}{2}$ inch in diameter and 12 inches long weighing 0.668 pounds makes 100 vibrations in 163.8 seconds. What is the radius of gyration of the body and of its cross-section?

Ans. k of cross-section is 1.127 inches.

3. In Problem 2 what is the moment of inertia of the cross-section?

Ans. 6.99 inches.^{4}

APPENDIX A

RIVETED JOINTS OF MAXIMUM EFFICIENCY

It is possible to design a riveted joint which has the same strength in tension, compression, and shear. Such designs frequently lead to unusual sizes of rivets or to inconvenient lengths of pitch. The American Railway Engineering Association specifies only ¾-inch, ⅞-inch, and 1-inch rivets for structural work. Larger and smaller rivets are used in boilers. In structural work, the pitch is required to be at least three times the diameter of the rivet. Smaller pitch is permitted in boilers. The pitch must not be too great in the row of rivets adjoining the edge of a plate which is to be calked. For this reason, multiple-riveted butt joints usually have one wide and one narrow butt strap, as shown in Figs. 48 and 49.

Formulas for joints of maximum efficiency are useful for deriving the approximate values of the pitch and the size of the rivets, and for showing the maximum efficiency which is possible with any given type of joint.

In the discussion which follows, $s_c =$ the compressive strength, $s_s =$ the shearing strength of the rivets, and $s_t =$ the tensile strength of the plates. Usually these are ultimate strengths. The results, however, are the same if the letters are used to mean allowable unit stress.

Ratio of Thickness of Plate to Diameter of Rivet.—If all the rivets are in single shear,

$$\frac{\pi d^2 s_s}{4} = dts_c, \tag{1}$$

in which d is the diameter of the rivet and t is the thickness of the plate.

$$t = \frac{\pi s_s}{4s_c} d. \tag{2}$$

When s_c is 95,000 and s_s is 44,000, as specified by the A.S.M.E. Boiler Code, Equation (2) becomes for single shear

$$t = \frac{0.7854 \times 44{,}000}{95{,}000} d = 0.3638d. \tag{3}$$

If all rivets are in double shear, the shear area is doubled and the bearing area is not changed. Consequently the thickness is doubled. For double shear,

$$t = 0.7275d \tag{4}$$

When part of the rivets are in single shear and part in double shear, as in Figs. 48 and 49, the bearing stress at the inner rows, which are in double shear, reaches its allowable value when the shearing stress is at its allowable value. In the rows which are in single shear, the shearing stress is the same, while the bearing stress is at only one-half of its allowable value. The rivets in double shear, therefore determine the thickness of the plate. For double shear combined with single shear

$$t = 0.7275d. \tag{4}$$

(If $s_c = 30{,}000\pi = 94{,}248$ pounds per square inch, and $s_s = 45{,}000$ pounds per square inch, then for single shear $t = \frac{3}{8}d$, and for double shear $t = \frac{3}{4}d$.)

Efficiency of Joint with Single Row of Rivets.—When the compressive strength is equated with the tensile strength of the net section,

$$s_ctd = s_tt(p - d), \tag{5}$$

in which t is the thickness of the plates and p is the pitch.

$$p = \frac{s_c + s_t}{s_t}d. \tag{6}$$

When the strength of the net section is equal to the strength of the joint, the efficiency is the ratio of the net width to the gross width.

$$\text{Efficiency} = \frac{p - d}{p} = 1 - \frac{d}{p} = 1 - \frac{s_t}{s_c + s_t} = \frac{s_c}{s_c + s_t}. \tag{7}$$

When $s_t = 55{,}000$ and $s_c = 95{,}000$, Equation (7) becomes

$$\text{efficiency} = \frac{95{,}000}{95{,}000 + 55{,}000} = \frac{19}{30} = 63.3 \text{ per cent.}; \tag{8}$$

$$p = \frac{150}{55}d = \frac{30}{11}d = 2.727d. \tag{9}$$

These equations apply to a lap joint with a single row of rivets or to a butt joint with one row on each side.

Efficiency of a Double-riveted Joint.—When there are two rows of rivets equally spaced, there are two rivets in compression in unit width.

$$2s_c td = s_t t(p - d), \tag{10}$$

$$p = \frac{2s_c + s_t}{s_t} d, \tag{11}$$

$$\text{efficiency} = \frac{2s_c}{2s_c + s_t}. \tag{12}$$

With the ultimate strengths of the A.S.M.E. Boiler Code,

$$p = \frac{190{,}000 + 55{,}000}{55{,}000} d = \frac{49}{11} d = 4.454d. \tag{13}$$

$$\text{Efficiency} = \frac{190}{245} = \frac{38}{49} = 77.5 \text{ per cent.} \tag{14}$$

These equations apply to a lap joint with a double row of rivets or to a butt joint with two equal rows on each side.

Efficiency of a Triple-riveted Joint.—When there are three rows of rivets, all equally spaced,

$$p = \frac{3s_c + s_t}{s_t} d = \frac{68}{11} d = 6.182d. \tag{15}$$

$$\text{Efficiency} = \frac{3s_c}{3s_c + s_t} = \frac{57}{68} = 83.8 \text{ per cent.} \tag{16}$$

Efficiency of a Triple-riveted Lap Joint with Long Pitch Twice the Short Pitch.—In a triple-riveted lap joint, Fig. 47, with the pitch of the outer rows twice as great as that of the inner rows, there are four bearing areas to be considered with the long pitch

$$4s_c td = s_t t(p - d), \tag{17}$$

$$p = \frac{4s_c + s_t}{s_t} d = \frac{87}{11} d = 7.901d.$$

$$\text{Efficiency} = \frac{4s_c}{4s_c + s_t} = \frac{76}{87} = 87.4 \text{ per cent.} \tag{18}$$

With Equation (18) it is necessary to investigate the combination of the strength of the net section at the middle row with the strength of the outer rivet in shear or compression. The net width at the outer row is $6.90d$ and at the middle row is $5.90d$. The ratio of the tensile strength of the net section at the middle row to the strength of the net section at the outer row is $^{59}/_{69}$, which is 0.85. Since only three-fourths of the total stress must be carried by the net section at the middle row, the strength is ample.

Problem

1. Two $\frac{3}{8}$-inch plates are united by 1-inch rivets in three rows to form a lap joint. The long pitch is $7\frac{7}{8}$ inches and the short pitch is $3\frac{15}{16}$ inch. Find the strength and efficiency of the joint.

Ans. Weakest in Shear. Efficiency $= \frac{138{,}232}{162{,}421} = 85.1$ per cent.

The joint of Problem 1 has the plates slightly thicker than one which would be equally strong in every way. Its tensile and bearing strength is, therefore, a little greater than its shearing strength, and its efficiency is somewhat lower than that of the ideal combination.

Efficiency of a Quadruple-riveted Lap Joint with Long Pitch Twice the Short Pitch.—In a quadruple-riveted lap joint which has the two middle rows of the same pitch and the two outer rows of twice this pitch the bearing strength of six rivets must be considered with the tensile strength of the net section at the outer row.

$$6\ s_c t d = s_t t(p - d), \tag{19}$$

$$p = \frac{6s_c + s_t}{s_t} d = \frac{125}{11} d = 11.36d. \tag{20}$$

$$\text{Efficiency} = \frac{6s_c}{6s_c + s_t} = \frac{114}{125} = 91.2 \text{ per cent.} \tag{21}$$

The ratio of the strength at the short pitch to the strength at the long pitch is $\frac{9.36}{10.36}$, which is considerably more than $\frac{5}{6}$. It is not necessary, therefore, to combine the tensile strength at the net section at the second row with the shearing or bearing strength at the outer row for any joint which approximates the ideal design.

Efficiency of a Double-riveted Butt Joint with All Rivets in Double Shear and with Long Pitch Twice the Short Pitch.—There are three bearing areas to be considered with the long pitch. The problem, therefore, is the same as that of the triple-riveted joint with all rivets equally spaced. Equation (16) gives the efficiency and Equation (15) gives the ratio of the long pitch to the diameter of the rivet.

Efficiency of Double-riveted Butt Joint with Long Pitch Twice the Short Pitch and with Outer Rows in Single Shear.—This is the joint shown in Fig. 48. When the shearing stress reaches its allowable value, the bearing stress in the outer rivet is only

one-half as great as in the inner rivets. Therefore, only two and one-half rivets in compression must balance the tension in the outer net section.

$$2.5\ s_c t d = s_t t(p - d). \tag{22}$$

$$\text{Long pitch} = \frac{2.5 s_c + s_t}{s_t} d = \frac{117}{22} d = 5.32d. \tag{23}$$

$$\text{Efficiency} = \frac{2.5 s_c}{2.5 s_c + s_t} = \frac{95}{117} = 81.1 \text{ per cent.} \tag{24}$$

The net width of long pitch is $4.32d$ while that of the short pitch is $3.32d$.

$$\frac{3.32}{4.32} = 0.768.$$

Since there are five shear areas in a unit width, and only one of these in the outer row, the net section at the inner row should be four-fifths as strong as that at the outer row. This joint designed by Equation (23) would fail by tension at the inner row and shear in the outer rivet. The efficiency would be less than 81.1 per cent. Since $0.80 \div 0.768 = 1.04$, the net section at the middle row may be made sufficiently strong if the thickness is made 4 per cent. greater than required by Equation (4). This increase of thickness, however, will lower the efficiency about 3 per cent. The thickness of the plates may be determined by Equation (4) and the net width at the inner row may be increased 4 per cent. Four per cent of 3.32 is 0.13. The net width of the inner row becomes $3.45d$ and the pitch becomes $5.45d$. With this adjustment, the gross section is increased a little more than 2 per cent., and the reduction of efficiency smaller. Problem 2 represents a joint which meets these conditions. Of course, no such joint would be designed by any engineer, as it would be impossible to get plates of this exact thickness. The problem is interesting as an illustration of the principles. Problem 3 is a practical joint which approximates the theoretical design.

Problems

2. A joint similar to Fig. 48 is made of plates 0.7275 inch thick, united by 1-inch rivets. The long pitch is 5.45 inches. Find the strength and efficiency of the joint.

	Pounds
Tension gross section	218,068
Tension net section outer row	178,055
Tension net section inner row	138,04[illegible]

Ans. Tension inner row and shear one rivet.................. 172,601
Compression two rivets and shear one rivet.............. 172,772
Shear in five sections................................ 172,790

$$\text{Efficiency} = \frac{172{,}601}{218{,}068} = 79.1.$$

3. Solve Problem 2 if the plates are ¾ inch thick and the long pitch is 5⅜ inches.

Ans. Weakest in shear. Efficiency = 77.9 per cent.

It is possible to design a double-riveted butt joint with the outer rows in single shear and the inner rows in double shear which is exactly as strong in tension at the inner row combined with shear at the outer rivet as it is in tension at the outer net section. This may be done if the pitch of the outer row is not exactly twice that of the inner row. Let m be the number of rivets in the inner row in a space equal to the pitch of the outer row.

$$(m + 0.5)\, s_c t\, d = s_t t(p - d)$$

$$p = \left(\frac{(m + 0.5)s_c}{s_t} + 1\right)$$

$$p - d = \frac{(m + 0.5)s_c}{s_t} d$$

$$p - md = \left(\frac{(m + 0.5)s_c}{s_t} + (1 - m)\right) d$$

Shear area $= (2m + 1)$ area of one rivet.

Shear taken by inner row $= \dfrac{2m}{2m + 1}$ part of total

$$\frac{2m}{2m + 1} = \frac{\dfrac{(m + 0.5)s_c}{s_t} + 1 - m}{\dfrac{(m + 0.5)s_c}{s_t}} = \frac{(m + 0.5)s_c + (1 - m)s_t}{(m + 0.5)s_c}$$

$$\frac{1}{2m + 1} = \frac{(m - 1)s_t}{(m + 0.5)s_c}$$

$$(m + 0.5)95 = (2m + 1)55\ (m - 1)$$

$$22m^2 - 30m - 20.5 = 0$$

$$m = \frac{30 + \sqrt{2{,}682}}{44} = 1.86 \text{ nearly}$$

If there are 8 rivets in the outer row to 15 in the inner row, this condition is nearly satisfied.

The efficiency is $\dfrac{(1.86 + 0.5)95}{(1.86 + 0.5)95 + 55} = 80.3$ per cent.

Triple-riveted Butt Joint with Two Rows in Single Shear.—When the pitch of the outer rows is four times that of the inner rows and the pitch of the intermediate rows is twice that of the inner rows there are four rivets in double shear and three rivets in single shear in a width equal to the long pitch.

$$5.5\ s_c td = s_t t(p - d) \tag{25}$$

$$p = \frac{(5.5s_c + s_t)}{s_t} d = \frac{231}{22} d = 10.5d. \tag{26}$$

Net width outer row $= 9.5d.$
Net width inner row $= 6.5d.$

$$\frac{6.5}{9.5} = \frac{13}{19}$$

Of the total load carried by a strip of width p, three-elevenths is transferred to the butt straps at the first two rows and eight-elevenths remains to pass through the third row.

$$\frac{8}{11} \div \frac{13}{19} = \frac{152}{143} = 1.062,$$

A little over 6 per cent. must be added to the net section at the inner row to prevent failure by tension at this section combined with shear in the outer rows.

The Boiler Code Committee of the American Society of Mechanical Engineers has worked out the efficiency of a number of joints. These are found in the *Transactions* of 1914, pages 1067 to 1075 and in the Report of The Boiler Code Committee, pages 103 to 109.

The design of riveted joints for structural work is treated fully in works on the "Design of Framed Structures."

See "The Design of Riveted Butt Joints" by A. A. Adler and discussion by S. F. Jeter, Transactions of American Society of Mechanical Engineers.

APPENDIX B

CALCULATION OF STRESS WHEN THE BENDING MOMENT IS NOT IN PLANE OF A PRINCIPAL MOMENT OF INERTIA

In Article 68, a rule was given for finding the stress in a beam when the bending moment was not in the plane of a principal moment of inertia. Although this rule is nearly self-evident, some readers may desire a proof.

Figure 253 represents any section of a beam for which XX and YY are the principal axes of inertia. The line MM, at an angle with YY, is in the plane of the bending moment, and the line BC, at an angle θ with XX, is the neutral axis. The element dA, whose coördinates with respect to the *principal axes are* x and y, and whose distance from the *neutral axis* is v, is subjected to a stress which varies as v.

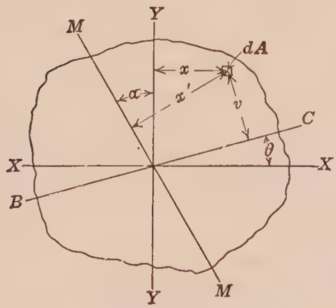


FIG. 253.—Moment at angle with principal axis.

$$v = y \cos \theta - x \sin \theta. \tag{1}$$

$$s = kv = k(y \cos \theta - x \sin \theta). \tag{2}$$

Since the external moment is in the plane MM, the resisting moment must lie in the same plane, therefore, the sum of the moments of all the stress on the entire area about the line MM must be zero. The perpendicular distance from dA to the line MM is

$$x' = y \sin \alpha + x \cos \alpha. \tag{3}$$

$$\int sx'dA = 0 = k\int vx'dA, \tag{4}$$

$$\int y^2 \cos \theta \sin \alpha \, dA + \int xy \cos \theta \cos \alpha \, dA -$$

$$\int xy \sin \theta \sin \alpha \, dA - \int x^2 \sin \theta \cos \alpha \, dA = 0; \tag{5}$$

$$I_x \cos \theta \sin \alpha - I_y \sin \theta \cos \alpha = 0, \tag{6}$$

in which I_x is the moment of inertia with respect to the axis XX, and I_y is the moment of inertia with respect to YY. The second and third terms of (5) include the product of inertia with respect to the principal axes, which is zero (see Article 205).

$$\text{Tan } \theta = \frac{I_x}{I_y} \tan \alpha. \tag{8}$$

Example

A 6-inch by 8-inch beam is subjected to a load perpendicular to its length making an angle of 30 degrees with the plane of the 8-inch faces. Find the angle between the neutral axis and the planes of the 6-inch faces.

When the line through the center parallel to the 6-inch faces is taken as the X axis,

$$I_x = \frac{6 \times 8^3}{12} = 256,$$

$$I_y = \frac{8 \times 6^3}{12} = 144.$$

$$\text{Tan } \theta = \frac{256}{144} \times 0.5774 = 1.0264,$$

$$\theta = 45° \, 45'.$$

The neutral axis makes an angle of 15 degrees 45 minutes with the line normal to the bending moment.

From Fig. 253 the component of the bending moment perpendicular to the neutral axis is $M \cos (\theta - \alpha)$. The moment of inertia with respect to the neutral axis is $I_x \cos^2 \theta + I_y \sin^2 \theta$, and $v = y \cos \theta - x \sin \theta$.

$$s = \frac{M \cos (\theta - \alpha)\, v}{I_x \cos^2 \theta + I_y \sin^2 \theta}; \tag{9}$$

$$s = \frac{M(\cos \theta \cos \alpha + \sin \theta \sin \alpha)(y \cos \theta - x \sin \theta)}{I_x \cos^2 \theta + I_y \sin^2 \theta}; \tag{10}$$

$$s = \frac{My(\cos^2 \theta \cos \alpha + \cos \theta \sin \theta \sin \alpha) - Mx(\cos \theta \sin \theta \cos \alpha + \sin^2 \theta \sin \alpha)}{I_x \cos^2 \theta + I_y \sin^2 \theta}. \tag{11}$$

$$My\,(\cos^2 \theta \cos \alpha + \cos \theta \sin \theta \sin \alpha) = My \cos \alpha\,(\cos^2 \theta + \cos \theta \sin \theta \tan \alpha) = My \cos \alpha\,\left(\cos^2 \theta + \frac{I_y}{I_x} \sin^2 \theta\right); \tag{12}$$

$$\frac{My(\cos^2 \theta \cos \alpha + \cos \theta \sin \theta \sin \alpha)}{I_x \cos^2 \theta + I_y \sin^2 \theta} = \frac{My \cos \alpha}{I_x}. \tag{13}$$

In a similar way the second part of (3) may be shown to be

$$\frac{Mx \sin \alpha}{I_y},$$

and

$$s = \frac{My \cos \alpha}{I_x} - \frac{Mx \sin \alpha}{I_y} \tag{14}$$

To find the fiber stress at any point in a beam when the bending moment is inclined to the principal axes of inertia, resolve the bending moment (or the applied forces) perpendicular to the two axes and compute the stress for each component separately. The actual unit stress is the sum of the results of these two, taken with the proper sign.

APPENDIX C

CAST-IRON BEAMS

Cast iron differs from most structural materials in that the ultimate strength in compression is many times greater than the ultimate strength in tension. A good sample of cast iron may have an ultimate tensile strength of 25,000 pounds per square inch and an ultimate compressive strength of 100,000 pounds per square inch. Working stresses in cast-iron members subjected to bending should be about 3,000 pounds per square inch tension and 10,000 pounds per square inch compression. Since the ultimate strength in compression is four times as great as the ultimate strength in tension, it is natural to suppose that this ratio should hold in the design of beams, and that the distance from the center of gravity of the section to the extreme compression fiber should be four times as great as the distance to the extreme tension fiber. However, since the neutral axis is shifted from the center of gravity of the section when the stress passes the elastic limit and since the stress in the tension fibers does not vary directly with the unit deformation, a lower ratio should be used.

A cast-iron beam of rectangular section, which was tested in bending, showed a modulus of rupture of 44,000 pounds per square inch. The same bar in tension broke under a load of 27,000 pounds per square inch. The shifting of the neutral axis and the deviation of the tension and compression diagrams from the straight line account for the difference.

Figure 254 gives the tension curve and part of the compression curve for specimens of cast iron which was tested at Watertown Arsenal. ("Tests of Metals," 1885, pages 475 to 490.) The tension curve is drawn from the mean of four tests and the compression curve from the mean of twelve tests. The averages in each case give points on a smooth curve although the readings from the individual specimens varied considerably.

The compression diagram is a straight line up to 13,000 pounds per square inch. The tension diagram begins to curve slightly

in the neighborhood of 3,000 pounds per square inch. The modulus of elasticity in tension is slightly greater than the modulus in compression up to about 9,000 pounds per square

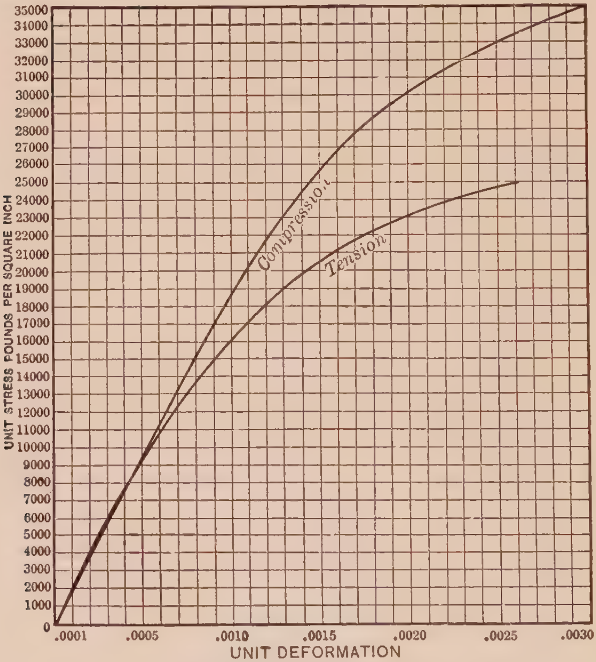


FIG. 254.—Stress-strain diagrams for cast iron.

inch. So long as the tensile stress in this cast iron does not exceed 3,000 pounds per square inch and the compressive stress does not exceed 10,000 pounds per square inch, the stress may be regarded as proportional to the distance from the neutral axis and the neutral axis is practically at the center of gravity of the section.

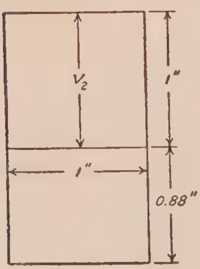


FIG. 255.

The problem of the unit stress at rupture is entirely different. Figure 255 represents a beam of rectangular section, which is 1 inch wide and is 1 inch high from the neutral axis to the upper fibers. The upper fibers are supposed to be in tension. It is desired to find the depth of the lower portion, which is in compression, in order that the total compressive stress shall equal the total tensile stress when the unit deformation in the top

fibers is 0.0025. From the curve, the unit stress in these outer fibers at the top is found to be 24,800 pounds per square inch. The area under the tension curve may be measured with a planimeter and the average ordinate may be determined by dividing this area by the base. If a planimeter is not available, the total area may be computed by dividing it into a number of strips. Each ordinate may be regarded as the mean altitude of a strip of width equal to the distance between the ordinates. For instance, the ordinate which corresponds with the unit elongation of 0.001 is 16,100 pounds per square inch. This may be taken as the altitude of a strip of width 0.0001 extending from elongation 0.00095 to 0.00105. The sum of the ordinates from 0.0001 to 0.0024, inclusive, multiplied by the width of the interval gives the area from 0.00005 to 0.00245. The half intervals from 0 to 0.00005 and from 0.00245 to 0.00250 may be computed separately. If the interval is taken as unity instead of 0.0001 to avoid decimals these areas are,

Area from 0 to 0.5	131
Area from 0.5 to 24.5 (sum of 24 ordinates)	398,650
Area from 24.5 to 25	12,350
Total	411,131
Total divided by 25 gives mean stress	16,445

If the width of the tension area is 1 inch and the height is 1 inch, the total tension is 16,445 pounds. If the width is b and the height is v_2 the area is bv_2 and the total tension is the product of this area multiplied by the mean stress.

The total compression below the neutral surface must be equal to the total tension above. If the width is unity as in Fig. 254 the total compression is the average compressive stress multiplied by v_1. Since this product is the area of the compression diagram, it is only necessary to find the ordinate which forms the right boundary of an area of 411,131 units below the compression curve. Since the diagram is a straight line up to the unit deformation of 0.00075, at which the unit stress is 14,300 pounds per square inch, this much of the area is a triangle.

Area from 0 to 7.5 (taken as a triangle)	53,625
Area from 7.5 to 20.5 (sum of 13 ordinates). . . .	309,570
Ordinate at 21.	30,850
Total to 21.5	394,045
Area required beyond 21.5	17,086
Total	411,131

When 17,086 is divided by 31,400, which is approximately the mean ordinate for the remainder of the area, the quotient is 0.56.

$$21.5 + 0.56 = 22.06.$$

The compression depth v_1 is to the tension depth v_2 as 22.06 to 25. The compression depth v_1 is practically 88 per cent. of the tension depth.

The moment of the total tension is obtained by multiplying each ordinate by its abscissa and adding the products. (The moment of the area between 24.5 and 25.0 is the product of half the ordinate at 24.75 multiplied by 24.75.) The result for a section one inch square above the neutral axis is 10,506 inch-pounds. In a similar way, the moment of the compression area, 1 inch wide and 0.88 inch deep, is 9,541 inch-pounds. The total moment is 20,047.

The maximum fiber stress in a beam 1 inch wide and 1.88 inches deep which is subjected to a bending moment of 20,047 inch-pounds, when calculated by the formula $S = \frac{Mc}{I}$, is 34,000 pounds per square inch. The actual tensile stress in the outer fibers is 24,800 pounds per square inch and the actual compressive stress is 31,400.

This calculation based on the experimental curves of Fig. 254 clearly indicates the difference which may exist between the modulus of rupture and the actual unit stress. Usually the difference is not so great, but, under some conditions it may be much greater than in this example.

If the beam were a T-section, the width of the flange and the thickness of the stem would have to be taken into account in the calculation of the total stress and the moment.

When it is not practicable to test full-size cast-iron beams, it is advisable to break smaller beams of similar sections. The modulus of rupture obtained experimentally from a beam of one section will apply to any beam of the same material of similar section.

APPENDIX D

DEFLECTION OF BEAMS BY THE EXACT FORMULA

In Chapter VIII, the equation of the elastic line of each beam was derived from the formula

$$M = EI\frac{d^2y}{dx^2}. \tag{1}$$

The complete expression for the moment is

$$M = \frac{EI\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}. \tag{2}$$

For such deflections as occur in ordinary structural beams, the approximate formula is sufficiently exact for all practical purposes, and is the form which is always used in the derivation of the elastic line of a beam or column. However, in order to gain some definite idea of the magnitude of the error in extreme cases of relatively large deflections, it is advisable to study a few examples of the integration of Equation (2).

The fundamental equation from which these expressions were derived is

$$M = EI\frac{d\theta}{dl}. \tag{3}$$

which may be written

$$M = EI\frac{d\theta}{dx}\frac{dx}{dl} = EI \cos\theta \frac{d\theta}{dx}. \tag{4}$$

The integral of Equation (4) is

$$EI \sin\theta = \int M dx + C_1, \tag{5}$$

which may be written

$$EI\frac{dy}{dl} = \int M dx + C_1. \tag{6}$$

The integral of Equation (1) is

$$EI\frac{dy}{dx} = \int M dx + C_1, \tag{7}$$

$$EI \tan\theta = \int M dx + C_1, \tag{8}$$

Equation (5) from the exact expression contains the sine of θ $\left(\text{or } \frac{dy}{dl}\right)$, while Equation (8) from the approximate expression contains the tangent of θ $\left(\text{or } \frac{dy}{dx}\right)$.

When the moment is constant, $\int M dx = Mx$. If the origin of coördinates is so chosen that $\theta = 0$ when $x = 0$, then $C_1 = 0$ and Equation (5) for constant moment is

$$EI\frac{dy}{dl} = Mx; \tag{9}$$

$$\frac{EIdy}{\sqrt{dx^2 + dy^2}} = Mx; \tag{10}$$

$$\left(\frac{EI}{M}\right)^2 dy^2 = x^2dy^2 + x^2dx^2; \tag{11}$$

$$dy = \frac{x\,dx}{\sqrt{\left(\frac{EI}{M}\right)^2 - x^2}} \tag{12}$$

$$y = -\sqrt{\left(\frac{EI}{M}\right)^2 - x^2} + C_2 \tag{13}$$

If the X axis is so chosen that $y = \frac{EI}{M}$ when $x = 0$. $C_2 = 0$ and

$$y^2 = \left(\frac{EI}{M}\right)^2 - x^2; \tag{14}$$

$$x^2 + y^2 = \left(\frac{EI}{M}\right)^2. \tag{15}$$

which is the equation of a circle of radius $\frac{EI}{M}$. This result agrees with Equation (4) of Article 70.

For a cantilever with a load P on the free end

$$M = -Px,$$

$$\int M dx = -\frac{Px^2}{2} + \left[C_1 = \frac{Pl^2}{2}\right]$$

when the origin is taken at the free end and l is the horizontal projection of the entire beam, and is not the original length.

$$\frac{EI\,dy}{\sqrt{dx^2 + dy^2}} = \frac{P}{2}(l^2 - x^2); \tag{16}$$

$$\left(\frac{2EI}{P}\right)^2 dy^2 = (l^2 - x^2)^2(dx^2 + dy^2); \tag{17}$$

$$\left(\left(\frac{2EI}{P}\right)^2 - (l^2 - x^2)^2\right)dy^2 = (l^2 - x^2)^2 dx^2; \tag{18}$$

$$\frac{dy}{dx} = \frac{l^2 - x^2}{\sqrt{\left(\frac{2EI}{P}\right)^2 - (l^2 - x^2)^2}}. \tag{19}$$

The expansion of the denominator of Equation (19) by the binomial theorem gives

$$\left(\frac{2EI}{P}\right)^{-1} + \frac{1}{2}\left(\frac{2EI}{P}\right)^{-3}(l^2 - x^2) + \frac{3}{8}\left(\frac{2EI}{P}\right)^{-5}(l^2 - x^2)^4 + \text{etc.} \tag{20}$$

When this series is multiplied by the numerator, $l^2 - x^2$, Equation (19) becomes

$$\frac{dy}{dx} = \frac{P}{2EI}(l^2 - x^2) + \frac{1}{2}\left(\frac{P}{2EI}\right)^3(l^2 - x^2)^3 + \frac{3}{8}\left(\frac{P}{2EI}\right)^5(l^2 - x^2)^5 + \text{etc.} \tag{21}$$

$$y = C_2 + \frac{Pl^2x}{2EI} - \frac{Px^3}{6EI} + \frac{1}{2}\left(\frac{P}{2EI}\right)^3\left(l^6x - l^4x^3 + \frac{3l^2x^5}{5} - \frac{x^7}{7}\right) + \frac{3}{8}\left(\frac{P}{2EI}\right)^5\left(l^{10}x - \frac{5l^8x^3}{3} + 2l^6x^5 - \frac{10l^4x^7}{7} + \frac{5l^2x^9}{9} - \frac{x^{11}}{11} + \text{etc.}\right) \tag{22}$$

The first three terms of the second member of Equation (22) are indentical with Equation (5) of Article 73. From these terms the constant $C_2 = -\frac{Pl^3}{3EI}$.

When the remaining terms which are given in Equation (22) are included,

$$C_2 = -\frac{Pl^3}{3EI} - \frac{1}{35}\left(\frac{P}{EI}\right)^3 l^7 - \frac{1}{231}\left(\frac{P}{EI}\right)^5 l^{11} - \text{etc.} \tag{23}$$

Example

A 6-inch by 1-inch wooden cantilever is 100 inches long and carries a load of 10 pounds on the free end. Find the deflection at the end if E is 2,000,000.

$$EI = 1{,}000{,}000, \quad \frac{P}{EI} = \frac{1}{100{,}000} = \frac{1}{10^5}, \quad l = 10^2.$$

$$C_2 = -\frac{10}{3} - \frac{1}{35} \times \frac{10^{14}}{10^{15}} - \frac{1}{231} \times \frac{10^{22}}{10^{25}} - \text{etc.};$$

$$C_2 = -3.333333 - 0.002857 - 0.000004 = -3.336294.$$

The second and third terms change the result only one part in 1,200.

If the load is raised to 50 pounds (which makes the maximum unit stress 5,000 pounds per square inch),

$$\frac{P}{EI} = \frac{1}{2 \times 10^4};$$

$$C_2 = -\frac{50}{3} - \frac{1}{35} \times \frac{10^{14}}{8 \times 10^{12}} - \frac{1}{231} \times \frac{10^{22}}{32 \times 10^{20}} - \text{etc.};$$

$$C_2 = -16.6667 - 0.3571 - 0.0135 = -17.0373$$

With this extreme deflection, the error of the ordinary equation is, apparently, a little more than two per cent.

Equation (23) would seem to indicate that the actual deflection of a cantilever with a load on the free end is greater than

$$y_{\max} = \frac{Pl^3}{3EI}. \tag{24}$$

This, however, is not the case. As ordinarily used l in Equation (24) is the total length of the beam. Since the moment arm x is the horizontal distance from the load to the section, the value of x at the fixed end (which has been represented by l in Equations (16) to (23) is the horizontal projection of loaded beam, and is less than the true length. In order to find the true deflection it is necessary either to express the moment in terms of the actual elements of length, or to find the actual total length in terms of the horizontal projection. The latter is the easiest, and will, therefore, be used.

For convenience, the element of length may be represented by ds instead of dl.

$$\frac{EIdy}{ds} = \frac{P}{2}(l^2 - x^2); \tag{16}$$

$$\left(\frac{2EI}{P}\right)^2 dy^2 = (l^2 - x^2)ds^2; \tag{25}$$

$$\left(\frac{2EI}{P}\right)^2 (ds^2 - dx^2) = (l^2 - x^2)ds^2; \tag{26}$$

$$ds = \frac{dx}{\sqrt{1 - \dfrac{P^2}{2EI}\quad(l^2 - x^2)}}; \tag{27}$$

$$ds = dx\left[1 + \frac{1}{2}\left(\frac{P}{2EI}\right)^2 (l^2 - x^2)^2 + \frac{3}{8}\left(\frac{P}{2EI}\right)^4 (l^2 - x^2)^4 + \text{etc.}\right] \tag{28}$$

$$s = [C = 0] + x + \frac{1}{8}\left(\frac{P}{EI}\right)^2\left(l^4x - \frac{2l^2x^3}{3} + \frac{x^5}{5}\right)$$
$$+ \frac{3}{128}\left(l^8x - \frac{4l^6x^3}{3} + \frac{6l^4x^5}{5} - \frac{4l^2x^7}{7} + \frac{x^9}{9}\right) + \text{etc.} \tag{29}$$

When $x = l$, the total length, which may be called l_i, is

$$l_i = l + \frac{1}{8}\left(\frac{P}{EI}\right)^2 l^5\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$
$$+ \frac{3}{128}\left(\frac{P}{EI}\right)^4 l^9\left(1 - \frac{4}{3} + \frac{6}{5} - \frac{4}{7} + \frac{1}{9}\right) + \text{etc.} \quad (30)$$

$$l_i = l + \frac{1}{15}\left(\frac{P}{EI}\right)^2 l^5 + \frac{1}{105}\left(\frac{P}{EI}\right)^4 l^9 + \text{etc.} \quad (31)$$

In the example above if the horizontal projection is 100 inches when the load is 10 pounds,

$$l_i = 100 + 0.0667.$$

If this length is substituted in Equation (24) instead of 100, the calculated result is

$$y_{\max} = 3.3400 \text{ inches.}$$

When the load is 50 pounds,

$$l_i = 100 + \frac{1}{15} \times \frac{10^{10}}{4 \times 10^8} + \frac{1}{105} \times \frac{10^{18}}{16 \times 10^{16}},$$
$$l_i = 100 + 1.667 + 0.059 = 101.726 \text{ inches.}$$

If this length is substituted instead of 100 inches in Equation (24), the result is

$$y_{\max} = 17.557 \text{ inches}$$

When Equation (23) is used to find the deflection of a cantilever which is loaded at the end, the length l is the horizontal projection of the loaded beam. If the original length is used, Equation 23 gives a greater error than Equation (24).

When a beam which rests on fixed supports is loaded at the middle, the moment arm of the vertical component of the reaction remains constant. The reaction has also a horizontal component which increases the deflection still further. The equation of the elastic line may easily be found from the approximate differential equation (Equation (1)). The solution from Equation (2) is evidently very difficult.

The formulas which have been derived in this appendix are sufficient to indicate the method of attack and also to show that the error of the usual equations is negligible, unless the deflections are extremely large.

Except wnen M is constant or $\frac{M}{I}$ is constant, the integration of Equation (2) gives a series. If the same moment equation is used with Equation (2) as is ordinarily used with Equation (1), (that is, if no allowance is made for the change in moment arms which is caused by the deflection) the first group of terms, which contain EI in the first power, are equivalent to the entire solution with Equation (1). The error which is caused by the use of Equation (1) instead of Equation (2) is generally smaller than that caused by the change in moment arms.

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